ECONOMETRICA

JOURNAL OF THE ECONOMETRIC SOCIETY

An International Society for the Advancement of Economic Theory in its Relation to Statistics and Mathematics

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VOL. 7, NO. 2 — APRIL, 1939

E C O N O M E T R I C A

Published quarterly by the Econometric Society
Colorado Springs, Colorado, U. S. A.

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Subscription Price $7.00 per annum. Single copies $1.75.

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Menasha, Wisconsin

HENRY SCHULTZ
1893–1938

ECONOMETRICA
VOLUME 7 APRIL, 1939 NUMBER 2

THE WORK OF HENRY SCHULTZ*

By HAROLD HOTELLING

AFTER A LIFE devoted with singular intensity and extraordinary success to the advancement of economic theory and its statistical complement, Henry Schultz was suddenly killed on November 26, 1938, while driving on a mountain road near San Diego. The accident was the more appalling because it means that he leaves no direct descendants to carry on his genius; for his wife and his two daughters, Ruth, aged 15, and Jean, aged 11, were also killed.

Henry Schultz was born in Russian Poland September 4, 1893. After attending the public schools of New York City and the College of the City of New York he received the Bachelor of Arts degree in 1916. In the same year he matriculated at Columbia University, where he studied with Professors Seligman, Seager, Moore, Mitchell, Willis, Mussey, Chaddock, Giddings, Dewey, and others. These studies were, however, interrupted in 1917 by the war. After the armistice he was awarded an army scholarship which enabled him to spend the spring and summer terms of 1919 at the London School of Economics and Political Science and at the Galton Laboratory of University College. During this time he attended the lectures of Cannan on economic theory, of Graham Wallas on political theory, of Hobhouse on philosophy, and of Bowley and Karl Pearson on statistics. Returning to the United States, he conducted various statistical and economic investigations for the War Trade Board, the United States Bureau of the Census, the United States Bureau of Efficiency, and the Institute of Economics. He was married to Bertha A. Greenstein in 1920.

When he completed his work for the doctorate at Columbia in 1925 he was director of statistical research in the Children's Bureau of the United States Department of Labor. His thesis dealt with "The Statistical Law of Demand as Illustrated by the Demand for Sugar," and was published in two installments in the *Journal of Political Economy*. Its total length was 81 pages, coming closer in this respect, as well as in method and spirit, to dissertations in the sciences than to those in

* This paper and that of Paul H. Douglas were read at a special joint session of the Econometric Society, American Economic Association, and American Statistical Association at Detroit, December 29, 1938. William Leonard Crum presided and spoke briefly of Henry Schultz.

the more verbose types of economics. In it he acknowledges the kindness of Isador Lubin, F. C. Mills, and W. C. Mitchell in reading the manuscript. His main inspiration was in the work of Henry L. Moore, whose pioneer attempts to derive demand curves from time series stirred his enthusiasm.

After entering the Department of Economics at the University of Chicago in 1926, Schultz gradually developed a large program of research of a kind definitely projected by his earlier studies. Beginning with sugar, he studied numerous commodities whose demand curves he attempted to obtain from the records of consumption and price. He also made some attempts to obtain supply functions, and shortly before his death discussed plans for extending his work in this direction. Convinced that economics could not become a science without careful quantitative studies, he examined one commodity after another with the primary object of exemplifying, checking, and extending economic theory. The neglect of theory so often evident in studies of particular industries and commodities was not a fault of Henry Schultz. While making numerous and extensive excursions into the forests of detail surrounding the consumption and production of the several commodities, he always regarded these excursions as means of purifying the representation of theory derived from his examination of the demand and supply schedules by removing or allowing for particular sources of disturbance.

Obtaining from the social-science research funds of the university the means for assistants and machines, Schultz trained a staff of workers, many of whom have since become prominent by their own work. He kept them busy searching out and correcting the data regarding the long list of commodities, drawing graphs, making calculations, translating relevant material, and otherwise building up the mass of results of which a large part appears in his final book. This book, *The Theory and Measurement of Demand*, which I reviewed in the *Journal of the American Statistical Association* for December, 1938, in some detail, is a massive achievement after which, apparently for the first time, Schultz paused in his research for a few months. He had left Chicago for a semester to teach at the University of California in Los Angeles while making a fresh start after the task that had absorbed him for so long. He jestingly remarked after the completion of this book that it was a good time to die.

At first Schultz used perforce the inefficient statistical methods then current, deriving demand functions from link relatives and percentage deviations from trend without adequate tests of significance in terms of probability. But as the light of modern developments of theory and

technique began to spread he was quick to utilize it. He made an intensive study of the great new work in mathematical statistics and used the results accurately and efficiently in his later investigations. He even made contributions to mathematical statistics himself. In this later work his usual procedure was to fit a regression equation in which one of the variates, price and quantity, would appear as a dependent and the other as an independent variate, with time and polynomials in time as additional independent variates. Sometimes the logarithms of prices and quantities would be taken as the variates. He called the result "the dynamic law of demand," and in a sense this appellation is appropriate. For any fixed value of the time variate these regression equations give approximations to the demand curves of classical theory, approximations whose accuracy is limited by the weaknesses of the *ceteris paribus* assumptions of that theory. From another standpoint, however, it will be observed that both cumulative and speculative effects are absent from Schultz's equations, which thus belong essentially to static rather than to dynamic economics, with time as a passive parameter, carrying along the gradually changing influences of a mass of unspecified sources of variation. His results do not, for example, offer an explanation of business cycles. He was aware of this, and preferred to study static theory, as more fundamental, rather than joining the great procession into monetary and business-cycle theory. He was extending the work of the classical economists in a different direction from that most popular at the moment.

But Henry Schultz's study of demand was not only statistical, nor was it even confined to the purely empirical. He studied general economic theory intensively, and probed deeply into the phases of this theory on which mathematics could throw a specially clear light. He worked through Pareto's extensive writings, which he greatly admired, though later observing certain weaknesses in them, such as Pareto's blunder in defining competing and complementary commodities by means of the ambiguous sign of the second derivative of a utility function not fully determined. He considered Pareto's most valuable work his forty-page article in the *Encyclopédie des Sciences Mathématiques*. He studied Cournot, Edgeworth, Cassel, Walras, Slutsky, and other mathematical economists thoroughly.

For the full study of economic theory and of statistical methods Schultz, like most economists, found his mathematical knowledge inadequate at first. In spite of his intense preoccupation with his teaching and economic research, he undertook while at Chicago to fill this gap, and actually did so to a remarkable extent. By reading, attending mathematics classes with young students while himself a professor,

and consultation with colleagues, he mastered advanced calculus and considerable areas of the theory of differential equations and the calculus of variations.

Relations between different commodities engaged Schultz's attention in his later years. In 1932 he read for the *Journal of Political Economy* a manuscript of mine on this subject, and made some suggestions, based on his study of Pareto, that resulted both in its improvement and in my starting on a new train of thought that eventually led to further results. Later he pointed out the important work of Slutsky published in Italian in 1915 in the same field. His discovery of this previously ignored work of Slutsky is an instance of the wide dragnet with which he sought out original ideas from all sources. The theory of demand for related commodities predicts under certain assumptions that relations of symmetry will be satisfied by the related demand functions. Schultz set out to determine from the data whether these relations are satisfied by actual demand functions. Though his results are somewhat inconclusive in this respect, owing to paucity of good data, they seem to indicate significant discrepancies in certain cases, and point the way to more definitive tests which will eventually become possible through the accumulation of data and the improvement of statistical methods.

Of his warm-hearted interest in individual students and other young economists and his encouragement of their researches it is impossible to give an adequate idea. This personal characteristic is dwelt upon again and again by former students and assistants. The following extracts from letters from them give some indication of his sympathetic understanding, of his strenuous devotion to work, and of his wide knowledge.

Harry Pelle Hartkeimer writes:

> I value my experience as his research assistant much more than the opportunity of being a student in his classes, for Professor Schultz was primarily a research worker. His courses were partial aspects of his research program and were never a survey of existing knowledge, which the students were expected to obtain themselves from the library. His courses were always a step forward into unexplored regions of statistical knowledge. This was evidenced by his frequent remark that he never repeated a course. He would say that next year he would have such and such article or book written (he was then working on it while teaching the course) and the students could then read the book, article, or monograph. It would no longer be necessary to give the course. At first he had no idea of the tremendous load this placed upon the average student who was taking other courses. Later on he did realize this somewhat, for he introduced at the University of Chicago the practice of allowing the student to register for double or triple credit in one course. Thus the student could take one course for 3 majors credit and devote his whole attention to the work under Professor Schultz, for 3 majors constitute full-time work for one quarter. Professor Schultz

worked from early in the morning until late at night in his office and when the school building was closed at 10 P.M. he took a large brief case of work home with him. He worked Saturdays and holidays the same as week days, and Sunday afternoons also found him in the office.

Professor Schultz was a careful student of current economic theory and he would frequently forecast developments in economic theory. I recall an incident in which he forecast a speech made by a president of the American Economic Association. He was at dinner with his colleagues who began to discuss forecasting. Professor Schultz smilingly remarked that he would not discuss general business forecasting but he would forecast the presidential speech which was to be given some days later. He mentioned that he did not know the president personally, he had not talked with him, or read the speech, but he was a careful student of current tendencies in economic theory. His forecast was amazingly definite and correct.

A letter from H. Gregg Lewis contains the following passages:

Whether in economics or statistics he began by building a mathematical-logical foundation of theory—a model to serve as a frame of reference for further analysis. He was an extremely able logician, widely read and trained in modern philosophy, mathematics, and the natural sciences. He impressed upon his students the necessity for thoroughly understanding all the premises upon which the logical model was built.

But he did not stop with the strict logical pattern; he proceeded to show us how we could test the assumptions underlying it, first by a priori reasoning—in which he exhibited his wide range of knowledge—and then by statistical techniques.

Schultz was an untiring worker, forever driving himself much more than any of his assistants. His own work day was a twelve-hour one or longer, and his work week, seven days.

But more than any of these, Mr. Schultz was to us an unselfish friend and adviser. He was always interested in all aspects of our personal well-being, and never in a patronizing way.

Jacob L. Mosak writes:

The fatherly interest which he showed in me was typical of the warm interest he had in all who were associated with him. He always gave freely of his time to advise, encourage, and aid promising students. Altogether he was one of the kindliest and most warm-hearted of persons whose friendship it was my privilege to enjoy.

He always welcomed, indeed insisted on, honest criticism, no matter how severe. He could discuss matters for hours and, as Professor Millis said, shed only light and no heat. His ripe judgment, open-mindedness, and fairness, his good taste and tactfulness commanded the respect and admiration of all who were associated with him.

Of Schultz's striking personal characteristics as seen by his colleagues at Chicago Professor Douglas writes in an interesting note. I shall conclude with a list of publications.

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HENRY SCHULTZ AS COLLEAGUE

By Paul H. Douglas

Since Professor Hotelling has discussed the larger scientific contributions of Professor Schultz, it is I believe appropriate for me to pay tribute to his worth as a colleague, a teacher, and a friend. Henry Schultz first joined us at The University of Chicago in the fall of 1926, twelve years ago. His vital inspiration had come from his work with Karl Pearson in London, and even more from his association with that lovable and nervous genius, Henry L. Moore, of Columbia. We had both been approximately contemporaneous students of Moore in the days from 1913 to 1917, but what was to me a more or less incomprehensible fog of formulae, which now and then lifted to let through the gleams of Moore's analysis, was to Henry pure gold. His work with Moore led him at once forward and backward. Forward into the field of building an inductive science of economics with the aid of mathematical analysis, and backward to the study and enjoyment of the great masters, Cournot, Walras, and, in his economic (though not sociological) writings, Pareto.

Nor were these dissociated but instead closely integrated interests. His study of the great economic pioneers of the past saved him from the blind empiricism which unfortunately has characterized so much of American statistical work, and enabled him to ask sensible questions which he then proceeded to try to solve. At the same time, his inductive work breathed fresh life into his analysis of the development of economic doctrine and rescued it from the empty and inconclusive formalism which it so often becomes. As scholar and problem solver he was, therefore, thoroughly integrated, and each section of his interests fed and enriched the other. He was similarly integrated in his happy combination of mathematical skill and economic insight. He may not have been a great mathematician, but he was certainly an accomplished one. He derived an aesthetic pleasure from the beauty and precision of mathematical analysis, and this was proper, but he also was always anxious to test the worth-whileness of this analysis in terms of its economic meaning. He, therefore, bound economics and mathematics into a unity, and was, I believe, with Irving Fisher and Henry Moore, the best example we have had in this country of a true econometrician. Similarly, his interests led him into the developments in the fields of physics, physical chemistry, and biology, where he commanded the respect of the specialists, and from which he brought back fresh insights and methods of attack.

To most of us, therefore, Henry Schultz was in his scientific work a powerful and integrated personality. For fifteen years, he devoted

his research energies to the measurement of elasticities of demand for a wide variety of commodities—a work as necessary to help make economics a more or less exact science as was the determination of atomic weights for the development of chemistry, and this finally found fruition in his masterly volume on *The Theory and Measurement of Demand*, which appeared only a few months before his death.

As a teacher, Professor Schultz was at once amazingly thorough, precise, and kindly. He heaped work in profusion upon his students, and insisted that they both do it accurately and know its significance. He was the very soul of kindness to the competent who were struggling against difficulties, and warmed them with his interest and personal devotion as he, too, had been warmed by his master, Moore. In his laboratories and in his research work, he trained a dozen or more of young men and women, now in their twenties or thirties, already winning their spurs, who I believe collectively have bright futures before them and who bid fair to carry on his tradition with high credit —among whom I should like to mention Wong, Wilcox, Hartkemeier, the McIntyres, Kellogg, Whitman, Friedman, Mosak, Bronfenbrenner, Samuelson, and Lewis.

As a colleague, Henry Schultz revived one's faith in the co-operative possibilities of scholarship. All too often our academic life is marred by the same blemishes of character which disfigure the world outside our walls. I am not at all certain that we as scholars are very much more open-minded than the great mass of our citizens, or that we are any less self-seeking, envious, or jealous; although we are perhaps less exposed to the temptations of dishonesty. But it was the supreme glory of our friend that these unlovely vices were totally absent from his nature, and that instead their opposite virtues were the dominant forces in his life. He was the truest seeker for truth I have ever known. He was never interested in trying to prove that he was right, and was thus never a verbal or logical point scorer or seeker after personal victory. He was only interested in finding out what was true. He, therefore, shed light upon every problem which he touched, and scarcely any heat. To such a man, the failure of an hypothesis to be confirmed by empirical data was never a tragedy or defeat, but merely the starting point for a new quest.

While Professor Schultz in his research and teaching did the work of at least two men, he was as interested in the problems which worried his colleagues as in his own. He would spend his time generously to understand what were the issues which were involved, what hypotheses should be set up, what tests employed, and what the statistical and economic meanings of the results actually were. There is no one from whom I personally learned as much, and I suspect that virtually all

of my colleagues found the same to be true in their own cases. He was, moreover, just as pleased when success came to others as to himself and, indeed, gave every evidence of being even more pleased. He was in fact modest almost to a fault, but underneath his modesty one always felt a rugged strength and an intellectual and moral independence which was like a rock.

Generous and unself-seeking, he was singularly forgiving. On several occasions he was the victim of personal and scientific attacks which were ugly in their tone, but he met these serenely and not only refused to bear their perpetrators any grudge but actually worked for their advancement.

His work, his family, and his friends absorbed the overwhelming share of his energies. He did not participate personally in current movements, deeming it best to husband his strength for the task which he believed to be his, and only his close personal friends knew where his personal sympathies lay. These were all on the side of upholding and furthering the essential rights and dignities of the great masses of humble human beings from whence he sprang. In politics he was, therefore, a progressive, and intellectually an ardent but tempered liberal. Perhaps because of the fact that he had been a front-line machine gunner during the war, he was for many years a quiet but determined pacifist. The aggressions of dictators and militarists finally convinced him, as they have so many others, that democracy cannot ultimately preserve peace and the dignity of human life by yielding further, and during the last summer he remarked to some of us that he was now ready to serve as a machine gunner again if it became necessary for us to check the spread of totalitarianism.

All in all, he was about the finest man it has ever been my privilege to know in academic life. The world of scholarship and of science (for I may so use the term in connection with him) was the richer for his fruitful life. We are much the poorer for his death at the full height of his powers.

University of Chicago

ANNUAL SURVEY OF STATISTICAL DATA: PARETO'S LAW AND THE INDEX OF INEQUALITY OF INCOMES

By C. Bresciani-Turroni

Pareto deserves a place of honor among those economists who have aimed "at a unification of the theoretical-quantitative and the empirical-quantitative approach to economic problems." His, sometimes misplaced,[1] exaltation of the inductive method led him to affirm that "the progress of economic science in the future will depend to a large extent upon the investigation of empirical laws, derived from statistics, which will then be compared with known theoretical laws, or will lead to the discovery of new laws. The new empirical laws will be essentially the result of the application to statistical data of the methods of interpolation."[2]

Pareto's empirical formula which describes the distribution of personal incomes seemed to open a wide and promising perspective of fruitful research. Professor Benini—who, following Pareto's suggestion, conceived in 1907 the bold plan of a new science of "inductive quantitative economics"—contended that Pareto, by stating his law of incomes, had written the first chapter of that new science.[3]

So far Pareto's and Benini's hopes have not been fulfilled. Though statistics and inductive verification have played in the last few decades a role of increasing importance, the progress of economic theory has been mainly achieved by a process of scientific investigation, which is by no means that suggested by Pareto in the passage quoted above. Interpolation has continued to be a useful statistical tool for practical purposes; but the empirical and quite arbitrary formulae, on which it is based, generally are lacking in scientific interest.

Pareto's Law forms an exception. The fact that incomes, in different places and epochs, are approximately distributed in accordance with this law, gives it a special significance. Though more than forty years have now elapsed since Pareto enunciated it, it is still attracting the attention of economists and statisticians.[3a] I only need recall the de-

[1] See Max Millikan, "Pareto's Sociology," Econometrica, Vol. 4, Oct., 1936, pp. 324–337.

[2] *Giornale degli Economisti*, May, 1907, p. 366.

[3] R. Benini, "Una possibile creazione del metodo statistico: L'Economia politica induttiva," *Giornale degli Economisti*, January, 1908.

[3a] See, for instance, G. F. Shirras, "The Pareto Law and the Distribution of Incomes," *Economic Journal*, December, 1935; D. H. MacGregor, "Pareto's Law," *ibid.*, March, 1936; B. P. Adarkar and S. N. Gupta, "The Pareto Law and Incomes in India," *ibid.*, March, 1936; N. O. Johnson, "The Pareto Law," *Re-*

tailed mathematical analysis which Bortkiewicz devoted to Pareto's law in a paper published in 1930[4] and the many applications of Pareto's formula to actual statistics that have been made in several countries in recent years.[5] The purpose of the present paper is to re-examine the celebrated chapter "La courbe des revenus" in Pareto's *Cours* in the light of the criticism to which it has been subjected in the last few decades.

THE RELATIONS BETWEEN THE AVERAGE INCOME, THE MINIMUM INCOME, AND THE INEQUALITY OF INCOMES

1. According to Pareto's well-known proposition, a rise in the minimum income or a diminution of the inequality of incomes is possible only when the aggregate income increases faster than the population (in other words, when the average income increases). Conversely,[6] when the income per capita increases, the lowest incomes will also rise or the distribution will become less uneven, or both these two effects will be produced in combination. Many economists have emphasized the practical importance of this proposition.[7] It seemed to afford a firm theoretical basis for an economic policy which considers the increase of production as the main problem[8] and refrains from any interference with the distribution process.

It is now a widespread opinion, that the loss of efficiency and the diminution of total income, which will probably be the consequence of any system of economic planning, would not be too high a price to

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view of Economic Statistics, February, 1937; Rufus S. Tucker, "The Distribution of Income among Income Taxpayers in the United States, 1863–1935," *Quarterly Journal of Economics*, August, 1938.

A selected bibliography of the distribution of incomes has been given by Professor Davis in Cowles Commission for Research in Economics, *Report of Fourth Annual Research Conference*, p. 21.

[4] "Die Disparitätsmasse der Einkommensstatistik," *International Statistical Institute*, Tokyo, 1930.

[5] These applications have sometimes led to noteworthy practical results. The most interesting example is that related by Lord Stamp in his book, *Wealth and Taxable Capacity*, 1922, p. 83. After having applied Pareto's formula to the supertax statistics, he told his colleagues "that they had missed over 1,000 payers in the lowest class, £5,000 to 10,000 . . . they promptly went and found them."

[6] Pareto observes (*Cours*, Vol. II, p. 322) that the converse proposition is not rigorously true, but that this qualification has, practically, little importance.

[7] See, for instance, Lanzillo, "La curva paretiana di distribuzione dei redditi," *Giornale degli Economisti*, 1932.

[8] "Free competition, since it stimulates production, indirectly contributes to the increase of the lower incomes and to the diminution of inequality among incomes." (Pareto, *Manuel*, p. 386.)

pay for the realization of the social ideal, which aims at diminishing the inequality of incomes. But this expectation would be frustrated, if Pareto's proposition were valid; for, if economic planning results in a diminution of the average income, the necessary consequence will be an increase in the inequality of individual incomes.

To Pareto's proposition was attached the prestige of mathematical methods. Pareto himself had declared that a demonstration of his proposition required the use of mathematics.[9] Its validity was not disputed even by mathematical economists such as Barone,[10] while "literary" economists, unfamiliar with mathematical methods, could not form a judgment upon it.

Professor Pigou's well-known criticism of Pareto's proposition centers in the following argument: it can by no means be proved that Pareto's Law is a law of even limited necessity and that any contemplated change must leave the form of income distribution unaltered.[11] Professor Pigou thus implicitly admits that Pareto's proposition would be true, if his law were valid for any time and place. Pigou's method of approach has been criticized a few years ago by Bortkiewicz.[12]

2. In order to understand the proposition correctly, it is necessary to bear in mind that, when the individual incomes are distributed according to either of the two principal forms which Pareto's Law assumes,[13] the average income X is given, respectively, by the following formulae: $X=\alpha h/(\alpha-1)$ and $X=(\alpha h+a)/(\alpha-1)$, where h is the lowest income.

In connection with the above formulae Pareto states two theorems. According to the first theorem, the minimum income h will vary directly with the average income, provided the degree of inequality among individual incomes does not vary, that is to say, when α and a do not vary.

Professor d'Addario has recently observed that Pareto here made a slip.[14] Indeed, on Pareto's own definition, inequality depends not only

[9] *Cours*, Vol. II, p. 320.

[10] See, "Nota sui prezzi politici ed economici" in Pantaleoni's volume, *La fine provvisoria di un epopea*, Bari, 1919, p. 57.

[11] *Economics of Welfare*, Fourth edition, 1892, p. 654.

[12] *Op. cit.*, p. 34. See also "Die Grenznutzentheorie als Grundlage einer ultraliberalen Wirtschaftspolitik" in *Schmoller's Jahrbuch*, 1898, No. 4, p. 119.

[13] Pareto's "first" law and "second" law are, respectively, $N_x=A/x^\alpha$ and $N_x=A/(x+a)^\alpha$ where N_x is the number of individuals with income equal to, or greater than x; and A, α, a are constants.

[14] *Intorno alla validità dei due teoremi paretiani sulla dinamica distributiva*, Roma, 1934, p. 6. D'Addario considers only Pareto's second law, but his objection is also valid with respect to the first law.

on α and a but also on h.[15] Therefore, the assumption that the degree of inequality is not altered implies that α and h in the first of the above formulae, and α, a, and h in the second formula, do not vary; but then the average income also will not vary. *Pareto's theorem becomes meaningless*, if we proceed from his definition of inequality.

However, we might ask what becomes of this theorem if we put other definitions of inequality. We may take as a measure of inequality for instance the "relative average deviation"[16] (that is to say, the absolute average deviation from the average income expressed in terms of the latter) and assume that the degree of inequality remains the same when the value of this index does not change. From the formulae expressing the relation between this index and Pareto's constants[17] it follows that:

[15] If N_h indicates the number of persons who receive an income equal to or greater than h, and N_x is the number of those who receive an income equal to or greater than x ($x>h$), and if we write

$$U_x = \frac{N_x}{N_h},$$

then, according to the definition of inequality given by Pareto in the *Cours*, p. 320, the greater U_x the less the inequality of incomes. In the case of Pareto's law in its two forms we have, respectively,

$$U_x = \left(\frac{h}{x}\right)^{\alpha} \quad \text{and} \quad U_x = \left(\frac{h+a}{x+a}\right)^{\alpha}.$$

Therefore U_x depends on α, a, h (x being supposed constant).

[16] The discussions as to the best single index of inequality have centered in Italy on the respective merits and the defects of the relative average deviation and of the concentration ratio (the well-known index proposed by Professor Gini, which measures the area between the Lorènz curve and the line of equal distribution). The mathematical properties of the two indexes have been analyzed by Professor Pietra ("Delle relazioni fra gli indici di variabilità," *Atti del R. Istituto Veneto*, 1914–15; "Note di statistica metodologica," in *Nuovi Problemi di Politica Storia ed Economia*, Ferrara, 1935). See also Ricci, "L'indice di variabilità e la curva dei redditi," *Giornale degli Economisti*, 1916; D'Addario, *Sulla misura della concentrazione dei redditi*, Roma, and "Un' indagine sulla dinamica distributiva," *Rivista italiana di sociologia*, 1930; Mortara, "Sulle disuguaglianze statistiche," XXII Session of the International Statistical Institute, London, 1934.

[17] The formulae are, indicating by D the relative average deviation, divided by 2:

$$D = \frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha}} \quad \text{and} \quad D = \left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1} \cdot \frac{h+a}{\alpha h+a},$$

according as the distribution follows Pareto's first or second law. (See Bresciani-Turroni, "Di un indice misuratore della disuguaglianza nella distribuzione della ricchezza," *Studi in onore di B. Brugi*, Palermo, 1910; Gini, *Variabilità e mutabilità*, 1912; Ricci, *op. cit.*) It can be easily shown that the value of D oscillates between 0 (perfect equality) and 1 (the maximum of inequality). The quantity D, equal to half the relative deviation, will be called in the present paper the "index of inequality."

(a) In the case of Pareto's first law—since the relative average deviation depends only on α—the average and the minimum income will vary in the same sense and in the same proportion when the degree of inequality of incomes is not altered;

(b) In the case of the second law, Pareto's theorem is not valid.

To give an arithmetical example, let us suppose four income distributions, which are characterized by the values of α, a, h, and X as given in Table 1.

TABLE 1

	α	a	h	X
I	1.5	1,000	500	3,500
II	1.6	1,966	500	4,610
III	1.6	1,573	400	3,688
IV	1.6	1,189	300	2,781

It can be easily verified that the "index of inequality" D (=half the relative average deviation) is equal to 0.495 for all four distributions. Comparing I with II we see that an increase in the average income leaves the minimum income unaffected. In III the average income is higher than in I while the minimum income is lower; but in IV both the average and the minimum income are lower than in I. To sum up: there is no uniform relation between the average and the minimum income.

The problem has, however, little practical importance. Indeed the "minimum income" has in Pareto's theorem a quite arbitrary meaning.[18] It is the minimum income with respect to that upper portion of the income curve, which may be represented by Pareto's formulae. In some cases it may be reasonably assumed that this "minimum income" is not far from the modal or most common income. The latter is an important statistical quantity, and it is unfortunate that actual statistics, being incomplete, generally do not allow of determining it. It would be very interesting to know its variations from place to place and from time to time, and to compare them with the variations of the average income; this would show whether the asymmetry of the income curve tends to increase or to diminish.

3. More important is Pareto's second theorem, which states a definite relation between the two fundamental indexes of economic welfare, namely, the national income (compared with the size of the population) and the way in which the national income is distributed among individuals or families.

[18] See Vinci, "Recenti vedute sulla legge di distribuzione dei redditi," in *Problemi di finanza fascista*, Bologna, 1937, p. 332.

According to Pareto, when h is constant a *general* diminution in the inequality of incomes takes place when α diminishes and a increases.[19] In this case the average income increases. Therefore a general diminution in the inequality of incomes will only occur when the average income increases.

This proposition is merely a consequence of the peculiar definition of inequality given by Pareto. Moreover, as Professor Bortkiewicz[20] has observed, the proposition does not presuppose Pareto's law, but is valid—on Pareto's definition of inequality—whatever might be the form of the income curve. Indeed, Pareto's proposition can be restated in the following more general form. Let $f_a(x)$ and $f_b(x)$ indicate the relative numbers of persons with an income smaller then x in two distributions I and II which range from the minimum income h to the maximum income k [so that $f_a(h)=f_b(h)=0$ and $f_a(k)=f_b(k)=1$)]; and let, for each other income x, $f_b(x)<f_a(x)$. It follows that the average income is greater in II than in I.[21]

But is Pareto's definition an appropriate tool for inquiring into the problem of income inequality? The answer now generally given—after the criticism made by Professor Benini and by several other Italian statisticians—is in the negative. We need a more adequate definition of inequality of income, such as that which underlies the relative average deviation or the concentration index. If the inequality of incomes is measured by either of these two indexes we are led to a conclusion which is exactly the opposite of that which results from Pareto's definition. A general diminution of the inequality of incomes takes place when α increases and a diminishes; *but then the average income diminishes.*

[19] Since on Pareto's definition the inequality of incomes diminishes when U_x increases, and since $U_x=[(h+a)/(x+a)]^\alpha$, it is easily seen by differentiating this function that U_x increases when α diminishes and a increases (*Cours*, Vol. II, p. 321).

[20] "Die Disparitätmasse . . . ," p. 34.

[21] As a matter of fact actual statistics show that a higher average income tends to be associated with a smaller percentage of income receivers below a given fixed limit x, irrespective of whether the income distribution follows Pareto's law. From the tables on pages 267 and 288 in Leven, *Income in the Various States*, 1925, I have calculated the following figures:

Per capita current income in each State (1919) (dollars)	Population depending upon family income smaller than 10,000 each (per thousand of whole population)
From 321 to 407	97.3
" 408 to 540	96.4
" 547 to 620	95.6
" 657 to 722	93.8
" 727 to 955	90.9

4. Prominent authors still attribute a paramount importance to Pareto's equation, as a description of actual income distributions.[22] The rather puzzling conclusion seems then to be—if we proceed from an appropriate definition of inequality—that any measure which aims at rendering the distribution of the national dividend more equal will injure the latter; and, conversely, that any measure which fosters the growth of the national dividend will render its distribution more unequal. However, on closer examination it is seen that the foregoing statement is unwarranted, even if we assume the validity of Pareto's law, for the two following reasons:

(a) The statement presupposes that the minimum income remains constant. But this is a quite unrealistic assumption, since a policy of "social justice" will aim not only at lowering the higher incomes, but also at raising the lower ones. As the above formulae show, the effect on the average income of an increase in α and of a diminution of a may be more than offset by an appropriate increase in h. Therefore, a policy which attempts to diminish inequality is perfectly compatible with measures taken to foster the growth of national wealth. But, Pareto's curve shows that a policy of social justice cannot be pushed too far, when it is not accompanied by an expansion of the national production and of the national dividend; since even a moderate increase of the lower incomes implies, if the average income does not rise at the same time, such a drastic cut of the higher incomes, that the most serious disturbances are likely to occur.[23]

There is the danger that mathematical formulae, such as those given by Pareto, oversimplify the problem and conceal some of its most interesting aspects. Pareto's formulae tell nothing about an important consequence of the redistribution of money incomes. Since the relative valuations of goods and of services would be altered, the diminution in the incomes of some members of the wealthy classes would entail the decrease of other high incomes—mainly of the incomes received by those who sell their services to the rich—so that a cumulative movement would set in and result in a vast destruction of money incomes. For this reason a redistribution of incomes, other things remaining equal, would probably be accompanied by a diminution of the average money income, so that still narrower limits than those resulting from Pareto's formulae are put to the increase of the lower incomes.

[22] See L. Amoroso, "Vilfredo Pareto," ECONOMETRICA, Vol. 6, Jan., 1938, p. 3.

[23] In a distribution which follows Pareto's first law, and where $h=500$, $\alpha=1.5$, and $N_h=1{,}000{,}000$, the maximum income (that is to say, that income k with respect to which $N_k=1$) is 5,000,000. If h is raised from 500 to 668 (the other small incomes will rise much less), and $\alpha=1.8$, the average income remaining the same, the maximum income is reduced to 1,439,000.

(b) We must distinguish, following Pareto, between *general* and *partial* changes in the distribution of incomes. A general diminution (increase) of inequality takes place when both parameters α and a of Pareto's curve vary simultaneously in the sense of a diminution (an increase) of inequality, that is to say, when α increases (diminishes), and at the same time a diminishes (increases). Then a general diminution (increase) of inequality is only possible, assuming h constant, when the average income diminishes (increases).

When both α and a increase (or diminish) together, inequality diminishes in some portions of the income curve, and increases in others. Pareto does not attempt to measure by a single index the total effect of such partial changes in the income distribution. It should be observed that the measures taken with a view to modifying the distribution of income will often bring about only partial changes and not a general diminution of inequality. For instance, a rise of moderate incomes will diminish the inequality between the latter and the higher incomes, but increase the inequality with respect to the small incomes.

It is easy to see that, when no general diminution (or increase) of inequality takes place, there is no uniform relation between the variations of the average income and the changes in the degree of inequality, as measured by the relative average deviation (or by the concentration index). I shall give an arithmetical example in Table 2.

TABLE 2

	h	α	a	Index of inequality D	Average income
I	500	1.55	1,600	0.5	4,318
II	500	1.6	1,930	0.494	4,550
II	500	1.5	800	0.484	3,100

If we pass from I to II we see that a diminution of inequality is accompanied by an increase in the average income; but from I to III inequality and average income decrease together.

5. As a matter of fact statistics point to a direct correlation between inequality and average income, when the "minimum" income is kept constant. A good example is given by the distribution of incomes and of properties in Prussian cities (1905)[24] shown in Table 3.

All income distributions from which the indexes of the above table have been calculated begin with the same "minimum" income (900

[24] The figures from which the indexes of inequality have been calculated will be found in Kühnert's article "Die Ergebnise der Einkommensteuerveranlagung nach Städtegruppen im Jahre 1905," *Zeitschrift des preussischen Stat. Bureaus*, 1907.

TABLE 3

Cities with population of	Incomes		Properties	
	Average (marks)	Index of inequality D	Average (marks)	Index of inequality D
less than 2,000	1,840	0.25	30,791	0.37
2,000– 5,000	2,085	0.29	37,162	0.42
5,000– 10,000	2,207	0.31	44,558	0.45
10,000– 20,000	2,183	0.32	51,047	0.48
20,000– 30,000	2,334	0.35	62,320	0.51
30,000– 50,000	2,066	0.33	65,292	0.51
50,000–100,000	2,454	0.38	87,911	0.53
more than 100,000	2,625	0.43	118,269	0.59

marks). When the minimum income varies, there seems to be no definite association between the average income and the inequality of individual incomes. An example is given by the German income statistics for 1920.[25] Their importance lies in that they cover a wide ground, since in that year 25.4 million heads of families or unattached persons were liable to the income tax. The law of May 29, 1920 did not fix any exemption limit, though it granted in certain specified cases some deductions and abatements. As a consequence, a large number of small incomes were subject to the tax: indeed, there were 10.4 million people with an annual income not greater than 5,000 paper marks (=about 490 gold marks). Data are available for each of the 18 German "Länder." Of these, Bremen and Hamburg show the highest average income (11,223 and 11,330 marks respectively) and at the same time the highest index of inequality (0.35 and 0.38). Some other "Länder" such as Lippe, Mecklenburg-Schwerin, and Mecklenburg-Strelitz, with low average incomes, show a low index of inequality. But on the whole there is but a loose connection between the average income and the inequality of incomes. In Table 4 the 18 Länder have been grouped into four categories, according to the size of the average income.

TABLE 4

Average income (marks)	Index of inequality D
7,220– 7,886	0.29–0.35
8,137– 8,766	0.29–0.34
9,017– 9,920	0.24–0.35
10,512–11,330	0.29–0.38

International comparisons also seem to confirm that there is no

[25] "Die deutsche Einkommenbesteuerung vor und nach dem Kriege," *Statistik des deutschen Reichs*, Vol. 312, Berlin, 1925.

definite relation between the size of the average income and the degree of inequality of personal incomes. In Italy both the average income and the inequality of incomes are less than in Great Britain. But in the United States, as Professor Young observes,[26] despite the relatively high average family income, incomes seem to be distributed with rather less inequality than in some of the other great industrial nations.

THE STABILITY OF THE INCOME CURVE

6. Another most interesting and much-discussed point is the statement that the actual values of the constant α—which measures the slope of the income distribution—fall within comparatively narrow limits.[27] They oscillate about 1.5 and the deviations from this value appear to be caused more by the imperfections of the statistical material than by some real cause. The income curve shows a striking stability in different epochs and places.

Pareto's conclusion was supported by the authority of Lord Stamp and Professor Bowley. The former contends that the slope of the income distribution remained practically the same in England throughout the nineteenth century, though there have been little shifts each way from time to time. All classes, although they may have become better off, have kept their relative positions and proportions with remarkable stability.[28] Professor Bowley also found that the increase in national dividend from 1880 to 1913 "was shared with remarkable equality among the various economic classes." He concluded that "the constancy of so many of the proportions and rates of movement found in the investigation seems to point to a fixed system of causation and has an appearance of inevitableness."[29]

The importance of Pareto's theory according to which the distribution of individual incomes is determined, not by the economic structure of society and by its institutions, but by the distribution of certain natural qualities inherent in men, has often been emphasized.

In this connection, I shall recall a paper published by Schmoller in

[26] "Do the Statistics of the Concentration of Wealth in the United States Mean What They Are Commonly Assumed to Mean?" *Quarterly Publications of the American Statistical Association*, Vol. 15, 1912.

[27] *Cours*, Vol. II, p. 312.

[28] *Wealth and Taxable Capacity*, p. 87, and *British Incomes*, p. 443. See also W. King, *The Wealth and Income of the People of the United States*, p. 92; H. McGregor, "Pareto's Law," *Economic Journal*, March, 1936; K.-G. Hagstroem, "La loi de Pareto et les réassurances," *Skandinavisk Aktuarietidskrift*, 1925; J. Marschak, *Elastizität der Nachfrage*, Tübingen, 1931, p. 84.

[29] *The Changes in the Distribution of the National Income, 1880–1913*, p. 27.

1895[30]—which seems to have escaped the attention of both the admirers and the critics of Pareto—where the German author develops a theory analogous to that of Pareto. It is a curious coincidence. Pareto's first essay on the income curve appeared in January, 1895,[31] a few months before the publication of Schmoller's paper. Afterwards Pareto reproduced in the *Cours* some of Schmoller's statistics, without, however, making any reference to the main thesis of the German economist.

As to Schmoller, surely his ignorance of the Italian language and his deeply rooted distrust of mathematical methods prevented him from being aware of Pareto's article.

The statistics quoted by Schmoller refer to the income distribution in Bâle, Frankfort, and Augsburg in the fifteenth century and in Saxony and Oldenburg in more recent years. He also compared the well-known Baxter table with King's figures. Unlike Pareto, who had given an analytical expression of the income curve in the article just quoted, Schmoller was content to compare the relative numbers of persons who received about equivalent incomes in various countries and epochs. He was struck by the stability of these percentages. Bâle and Frankfort showed in the nineteenth century about the same structure of the personal distribution of income as in the fifteenth century; and so England in 1688 and in 1867; and England in 1688 and Saxony in 1892. Schmoller was led to conclude that the differences in the personal qualities are the fundamental cause of the inequality among individual incomes; though he admitted the influence of chance, violence, economic evolution, and social institutions such as inheritance.

In Schmoller's opinion there is no uniform connection between the changes in the general level of incomes and their inequality. Schmoller's view, backed by a profound knowledge of economic history, is more realistic than Pareto's. Schmoller observes that, in periods of rapid increase of wealth, at the beginning it is especially the classes who lead the economic progress who get almost all the benefit of the material advance, so that the inequality of incomes tends to increase. But afterwards, new and better forms of social life are found, new social institutions are created, and a rise occurs in the economic standard—at the same time, in the moral and intellectual standard—of the working classes and of those other classes, who, owing to the

[30] "Die Einkommensverteilung in alter und neuer Zeit," *Jahrbuch für Gesetzgebung, Verwaltung und Volkswirtschaft*, 1895, p. 1067. Some considerations on the stability of the distribution of income among persons and its dependence upon the distribution of human natural faculties are also found in Engel's paper, "Die Klassensteuer und klassifizierte Einkommensteuer im preussischen Staate, 1852–1875," *Zeitschrift des königl. statistischen Bureaus*, Berlin, 1875, p. 127.

[31] "La legge della domanda," *Giornale degli Economisti*, Jan., 1895, p. 61.

more simple technical methods to which they clung, or to their being excluded from the market and from the sources of exceptional profits, had not shared hitherto in the increase of wealth.

Schmoller further observes that during periods of economic depression very often the inequality of incomes diminishes. He adds, however, that, if depression continues for a long time and economic progress stops entirely, the distribution of individual incomes is likely to become again more uneven; an example is afforded by the economic conditions in some countries of Europe towards the end of the sixteenth century and during the seventeenth century.

7. In a paper published many years ago[32] I pointed to the close association between the changes in the value of Pareto's parameter α and some definite economic movements. The value of α is generally lower in industrial than in agricultural regions; it is lower in rural than in urban areas; it regularly diminishes with the increase in the size of the cities; it diminishes during the expansion phase of the business cycle, whilst it again increases during the succeeding phase of depression. These conclusions have been confirmed by the researches of other writers, such as Gini, Savorgnan, and more recently Shiomi and the statisticians of the American National Bureau of Economic Research.[33] It was thus possible to get some insight into the nature and the causes of the changes in the distribution of incomes.

Probably the best statistics of a kind to show the distribution of income among individuals of a large country are the Prussian statistics.[34] They now cover a period of more than a century, since the oldest available figures are those for the year 1821. Of course it is material of very unequal statistical value; and the methods of assessment were changed several times.

For comparison purposes I have chosen the years 1821, 1854, 1873, 1875, 1892, 1913, 1918, and 1928. The original figures have been plotted on a double logarithmic diagram, Figure 1.[35]

[32] "Dell' influenza delle condizioni economiche sulla curva dei redditi," *Giornale degli economisti*, 1905.

[33] Gini, "Indici di concentrazione e di dipendenza," *Biblioteca dell'Economista*, Serie 5, Vol. 20, 1910; Savorgnan, *La distribuzione dei redditi nelle provincie e nelle grandi città dell'Austria*, Trieste, 1912, Museo Commerciale; Shiomi, "Die Entwicklung des Volkseinkommens in Japan in den Jahren 1903–1919," *Archiv für Sozialwissenschaft*, Vol. 53, 1925; National Bureau of Economic Research, *Income in the United States*, 1922.

[34] Taussig, *Principles of Economics*, 1925, Vol. 2, p. 254.

[35] Sources: for 1921, Neumann, "Zur Lehre von den Lohngesetzen," *Jahrbücher für Nationalökonomie*, 1892, Vol. 59; for 1854–1875, Engel's paper quoted above; for 1892, 1913, 1918 *Statistisches Jahrbuch des preussischen Staates;* for 1928, *Das Deutsche Volkseinkommen vor und nach dem Kriege*, published by the Statistical Office of the Reich, 1932.

The data for 1821 deserve but little confidence. Moreover they do not include some large cities, which paid other taxes instead of the income tax. Therefore the difference between the slopes of distribution in 1821 and 1854 is probably due to a large extent to the reform of 1851, which resulted in a more rigorous assessment of the higher incomes. Other German sources of information, however, also show that the distribution of income became more unequal in the first half of the nineteenth century.[35]

More reliable material is available since 1852. During the period covered by the chart considerable shifts have occurred in the slope of the distribution, and more especially from 1854 to 1873—in those

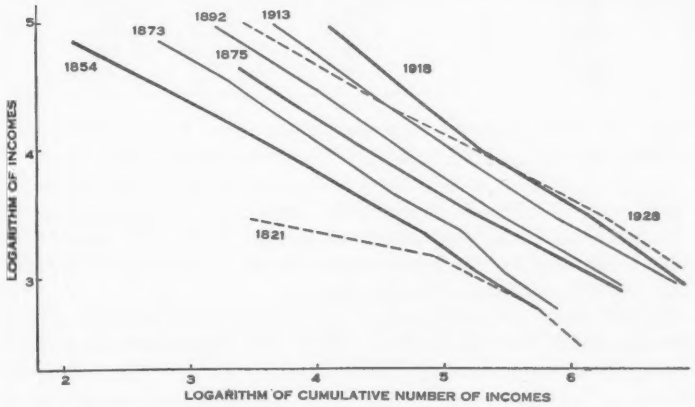

FIGURE 1.—Distribution of personal incomes in Prussia.

years modern "capitalism" developed in Germany—and from 1913 to 1918, during the Great War, which—unlike what happened in some other countries, for instance, in the United States—resulted in Germany in a sharp increase in the inequality of incomes. From 1873 to 1875—during the great crisis—and from 1918 to 1928 the slope of distribution again shifted the other way. It is not possible to measure the influence of the new methods of assessment, brought about by the new laws on income tax; but many other facts leave no doubt that the distribution of incomes has become less unequal in Germany as com-

The "minimum" income was 100 thalers in 1821 (a thaler was afterwards put equal to 3 marks); 200 thalers in 1854 and 1873; 220 thalers in 1875; 900 marks in 1892, 1913, and 1918; 1,200 marks in 1928.

[35] See Neumann's paper quoted above.

pared with the prewar situation. It is a most curious fact, revealed by the chart, that *in 1928 the slope of income distribution was about the same as in 1854*.

The values of α, which have been calculated according to the method indicated by Pareto,[36a] and those of the index of inequality are given in Table 5.

TABLE 5

	α	Index of inequality D
1821	2.45	—
1854	1.78	0.32
1873	1.51	0.38
1875	1.78	0.28
1892	1.52	0.37
1913	1.51	0.33
1918	1.35	0.37
1928	1.85	0.29

It will be seen that the variations in the value of α are substantial, and not merely of a kind to be only "detectable under a microscope."[37]

8. Nevertheless, the statistics I have so far examined include only a fraction of the Prussian population. Lord Stamp's statement as to the stability of the distribution of income in England also refers only to the "tail" of the income curve; and it is doubtful whether his conclusions are valid for the whole curve.

I have found for the year 1875 an estimate of the total income of Prussia and of its distribution among individuals (including those below the exempt amount). It was made by Engel.[38] The aggregate income of 11,572,413 heads of families or unattached persons amounted to 2458.9 million thalers; the average income was therefore 212 thalers (or 636 marks). The figures given by Engel show that 22.2 per cent of the total number of families received incomes above 220 thalers, and that the corresponding percentage of total income was 50.2. Therefore the index D is approximately equal to 0.28.

At about the same time (1878) the Statistical Office of Saxony published detailed figures on the national income of Saxony and its distribution among persons, taking into account also those below the exempt amount.[39] From these figures the value of D can be easily

[36a] "Il metodo di figurare i fenomeni economici," *Giornale degli Economisti*, 1896, p. 84.

[37] Snyder, "The Pareto Curve and its Significance for Our Time," Cowles Commission for Research in Economics, *Report of Third Annual Research Conference on Economics and Statistics*, 1937, p. 61.

[38] *Op. cit.*, p. 129.

[39] *Zeitschrift des sächsischen statistischen Bureaus*, 1878, p. 201.

calculated. It is about 0.35. The distribution was then more unequal in Saxony—an industrialized district—than in Prussia.

From 1892 to 1913 fairly homogeneous material is available for Prussia. For that period, and for the year 1928 also, it is possible to measure the inequality of incomes among the whole population thanks to the estimates of the aggregate income below the exemption amount, which have been recently made by the Statistical Office of the Reich.[40] I have calculated the indexes of inequality shown in Table 6.

TABLE 6

	Average income (marks)	Index of inequality D
1892	1,110	0.26
1896	1,107	0.26
1900	1,215	0.29
1907	1,355	0.30
1913	1,495	0.32
1918	1,827	0.39
1928	1,736	0.31

The foregoing table shows that from 1892 to 1918 the average money income of the German population increased—the rise was, however, offset in part by the diminution in the purchasing power of money—and that at the same time the distribution of income became more unequal. Account must be taken, however, of a disturbing factor. During the period under consideration, the number of individuals per family continually diminished. There has been an increasing tendency for members of working and middle-class families to become independent, so that their incomes were assessed separately from those of the heads of households, and the number of small incomes tended to increase. To what extent this shift affected the degree of inequality as measured by the index D is not possible to decide.

9. Another point might be worth noticing in this connection. Statisticians have always confined their attention to the distribution of income *received*. In this way a relevant fact is lost sight of, namely

[40] *Das deutsche Volkseinkommen*, 1932. These more recent data modify those given previously by the Statistical Office of Prussia in its yearbooks. The estimates of the latter are substantially lower. The detailed figures given in the *Mitteilungen aus der Preussischen Einkommensteuerstatistik* enable one to find out easily the percentage number of people with incomes above the average income and the corresponding percentage of total income. For 1928, however, these two percentages had to be calculated by interpolation. The index for 1918 has been calculated from the data published by the Statistical Office of Prussia (*Statistisches Jahrbuch für den preussischen Staat*, 1918).

I have the impression that the indexes of inequality, which result from the figures given by the Statistical Office of the Reich, are too low.

how incomes are spent. Since a large portion of the high incomes is absorbed by direct taxes, philanthropic contributions, and savings, obviously the distribution of income consumed is less unequal than the distribution of income received. Some interesting facts are revealed by a recent American inquiry made by Professor Moulton.[41] In 1929, in the United States, families with incomes under $1,500, and unattached individuals with incomes under $750, constituted 41 per cent of the population and accounted for 11.7 per cent of the aggregate income received, but for a substantially larger share (17.7 per cent) of the aggregate income spent on consumption goods. According to Professor Moulton savings were made for the most part by people with incomes above the average.

From the detailed figures on the distribution of incomes received which are found in Moulton's book, the value of the index of inequality D can be easily calculated. It is 0.382. On the other hand, on page 261 Professor Moulton gives the amount of living expenses in the various groups of income receivers. The average expense per family was $2,595. The number of families which spent more than this sum was about 33.6 per cent of the whole number, while their aggregate consumption was 61.8 per cent of the total income consumed, so that the value of D is 0.282. The difference between the two values of D is very considerable. Since the percentage which is saved increases when the national income is more unequally distributed, it is possible that an increase in the inequality of distribution of income received—such as happened in Prussia between 1892 and 1918—exerts but a small influence towards increasing the inequality of distribution of income consumed, or is even accompanied by a positive diminution of inequality in the latter sense.

Moreover, it should be observed that when the incomes of the wealthy classes increase, there is a tendency for these classes to pay higher prices for "quality" goods and for certain services, such as those of doctors, lawyers, and so on. Therefore new high personal incomes are created. It follows that the process towards a greater inequality in the distribution of income is to a certain extent self-inflammatory.

10. Reverting to the Prussian indexes of inequality, they confirm, even when the whole population is taken into account, that the distribution of income became probably less unequal between 1918 and 1928. On the whole, the distribution in 1928 was not very much altered as compared with 1875. Indeed, the shares of total income possessed by some given fractions of the population at the two dates were as in Table 7.

[41] *America's Capacity to Consume*, 1934.

TABLE 7

1875			1928		
(a)	(b)		(a)	(b)	
56.9	32.1		57.7	29.1	
31.0	30.0		31.7	32.6	
6.7 } 12.1	11.3 } 37.9		6.5 } 10.6	14.1 } 38.3	
5.4 }	26.6 }		4.1 }	24.2 }	
100.0	100.0		100.0	100.0	

(a) Percentages of the population. (b) Percentage of total income.

The percentage of total income held by the highest 10 per cent of the population has become somewhat larger in 1928 as compared with 1875; but on the whole the percentages shown in Table 7 are much the same in the two years.[42] In the meantime, however, the average income per family (or unattached person) rose from 636 to 1,735 marks;[43] while the Prussian population increased from 25 millions to 39 millions. It does not seem, from these results, that the "capitalistic" system has worked out so badly in Germany as its German critics now contend.[44]

11. It would be interesting to enter the promising field of international comparisons. I shall confine myself to a few figures concerning England. From the income statistics given by various authors I have calculated indexes of inequality as given in Table 8.

The table does not confirm Professor Bowley's statement that incomes were during the nineteenth century more equally distributed in England than in Prussia.[45] On the contrary England shows throughout the whole period much higher indexes of inequality than Prussia; which points to the persistency of some peculiar causes making for a greater inequality of incomes in the former country. An index of inequality nearly equal to the English index is found only in the Prussian cities with more than 100,000 inhabitants. Another feature of the English figures is—unlike what has been found for Prussia—the striking stability of the index of inequality. The first important change seems to

[42] The published data do not allow the examination of another interesting aspect of the income distribution, namely, the variations in the range of individual incomes (that is to say, the difference between the lowest and the highest incomes in the two epochs).

[43] According to J. Kuczynski's estimates the index of the cost of food rose in Germany from 103 to 195 between 1875 and 1928 (1900 = 100). A more comprehensive index is not available. (Kuczynski, *Löhne und Ernährungskosten in Deutschland, 1820 bis 1937*, Leipzig, 1937.)

[44] With respect to Saxony, the index of inequality D, calculated for the whole population, was 0.32 in 1928 as against 0.35 in 1878.

[45] Select Committee on the Income Tax, 1906, *Evidence*, p. 81.

TABLE 8

Year and Source	D
1801 (Stamp)[1]	0.47
1843 (Giffen)[2]	0.42
1850/51 (Levi)[3]	0.43
1867 (Baxter)[4]	0.41
1879/80 (Levi)	0.42
1913 (Connor)[5]	0.42
1924 (Connor)	0.36

[1] *British Incomes and Property*, p. 514.

[2] Giffen's series (distribution of assessments, Schedule D)—when plotted on a double logarithmic chart—is represented by an approximately straight line ($\alpha = 1.437$). From this value of α we have $D = 0.42$.

[3] Levi, *Wages and Earnings of the Working Classes*, London, 1885, p. 48.

[4] According to the well-known Baxter table (*National income*, London, 1868, p. 64) the average income per assessment or person was about £59. The exact number of people with income above the average income cannot be determined from Baxter's table. But assuming that all incomes of Class III (2) (average about £54) and that all incomes of class IV (average about £29) were below the general average, the index of inequality is 0.41. The correct index will be a little higher.

[5] L. R. Connor, "On Certain Aspects of the Distribution of Income in the United Kingdom, 1913 and 1924," *Journal of the Royal Statistical Society*, 1924.

have happened after 1913. Between 1913 and 1924 the index diminished from 0.42 to 0.36. It appears that after 1913 England's income has been approaching the less unequal distribution of other countries, such as Germany and the United States.

12. The examination of Prussian and English statistics leads to the conclusion that, if intervals of sufficient length are taken, there is without doubt a considerable stability in the degree of inequality of incomes. So far Pareto's theory is confirmed, and acquires a deep significance, since it points to the action of some deap-seated forces which tend to restore the fundamental form of the income curve after this form has been deflected towards a more pronounced, or a less pronounced, inequality. But very substantial changes in the degree of inequality may happen each way in shorter periods. This explains why different authors have drawn quite divergent conclusions, as to the trend of income distribution, from statistical data concerning the same country, but covering different time intervals.

Actually the index of inequality has been found to vary within very wide limits (the extreme values given in the foregoing tables are 0.26 and 0.59). Very distinct shifts have occurred in Prussia in periods of economic or political unrest. Hence Pareto's contention that the distribution of income is hardly affected by changes in the economic structure of the various countries, by political causes, or by juristic institu-

tions, is not confirmed by statistics. Assuming that the changes in the income distribution have but little importance and that it is the changes in the average income that matter, Pareto was inclined to conclude that economic policy ought to be much more concerned with the progress of wealth than with its distribution;[46] a conclusion which was in accordance with the traditional trend of economic thinking in Italy. But the problem is whether economic policy ought to lay more stress on the short-run, or on the long-run, effects. Sometimes the social effects of an acute increase in the inequality of incomes may be too important to be ignored; even if it might be expected that in the long run the economic forces will restore the social equilibrium, and that in the meantime that initial increase of inequality, but stimulating saving and the creation of new capital goods, will have brought about an increase in the national dividend. In any case, the problem of State interference with the distribution of personal incomes has a long-term and a short-term aspect; and the relative weight to be attached to each aspect will vary according to circumstances.

13. In Edwin Cannan's opinion the main object of the economic theory of distribution is to give an answer to the question "why inside each community some individuals and families are above and others are below the average in wealth."[47]

TABLE 9

Prussian cities with population:	Incomes		Properties	
	P	*I*	*P*	*I*
below 2,000	0.73	0.49	0.78	0.40
2,000– 5,000	0.75	0.46	0.77	0.36
5,000– 10,000	0.76	0.45	0.78	0.33
10,000– 20,000	0.78	0.45	0.79	0.31
20,000– 30,000	0.79	0.44	0.80	0.29
30,000– 50,000	0.81	0.48	0.80	0.28
50,000–100,000	0.81	0.43	0.79	0.26
above 100,000	0.84	0.41	0.82	0.23

If this view be accepted, it will be necessary to know above all the actual numbers of individuals above and below the average in wealth or income. In Tables 9 and 10, *P* indicates the ratio of the number of individuals with incomes below the average to the total number of income receivers, while *I* denotes the percentage of total income possessed by the people below the average income. (In several cases interpolations have been made by means of Pareto's First Law.)

[46] *Manuel*, p. 424.

[47] "The Division of Income," *Quarterly Journal of Economics*, 1905, p. 341.

According to Tables 9 and 10, the lowest and the highest value of P (for incomes received) are 73 and 84 per cent, respectively. Out of 31 values of P, recorded in the tables, 22 fall within the limits 0.78 and 0.82. A distinctly different value of P (0.66) is shown by the United States series of incomes consumed. Whether P is calculated taking into account the whole population or only the classes above the exemption limit does not substantially influence the result; for instance, the value of P is 0.82 in the English series of 1801, which refers to 320,759 taxpayers only and 0.79 in the Prussian series of 1913 which includes the whole population. The figures for the Prussian cities shows that P regularly increases with the size of the cities.

TABLE 10

	P	I
England, 1801	0.82	0.34
England, 1850/51	0.80	0.37
England, 1879/80	0.83	0.41
England, 1913	0.81	0.39
England, 1924	0.80	0.44
Saxony, 1878 (whole population)	0.79	0.43
Prussia, 1875 (whole population)	0.78	0.50
Prussia, 1875 (incomes above 220 thalers)	0.80	0.52
Prussia, 1902 (whole population)	0.84	0.58
Prussia, 1908 (whole population)	0.80	0.50
Prussia, 1913 (whole population)	0.79	0.47
Prussia, 1913 (incomes above 900 marks)	0.82	0.49
United States, 1928, incomes received	0.75	0.36
United States, 1928, incomes consumed	0.66	0.38
Italy, 1929 (schedule B)	0.78	0.38

But P appears to be much less variable than I and the index of inequality D (which, it will be remembered, is the difference between P and I). Indeed, in Tables 9 and 10, the extreme values of I are 23 and 58 per cent, and those of D are 25 and 59 per cent. It will be seen in many cases that the variations of D are due much more to the influence of I than to that of P. For instance, in the Prussian cities with more than 100,000 inhabitants, the value of P is 0.84 with respect to incomes, and 0.82 with respect to properties, while I is equal to 0.41 and to 0.23 respectively. In England the value of P was much the same in 1801 and 1924; and the difference in D is explained by the fact that the percentage of total income possessed by the same percentage P of taxpayers increased in 1924 as compared with 1801. In 1913 the percentage of income receivers with an income below the average was much the same in the United Kingdom and in Prussia, but the percentage of total income which they possessed was substantially greater in Prussia than

in the former country; and this accounts for the lower Prussian index of inequality.

The fact, shown by the foregoing tables, that the percentage P undergoes comparatively slight changes is an interesting aspect of the stability of the income distribution. But it should be observed that the percentage P has, from its arithmetical nature, little sensitiveness to changes in the income distribution. When, for instance, all individual incomes are increased, or diminished, by the same absolute amount, P will be unaffected, while I will increase in the former case and diminish in the latter. Therefore the percentage P is not a trustworthy index of inequality, if the definition underlying the relative average deviation (or other indexes based upon a similar concept of inequality) be accepted.[48]

For Professor Allyn Young, on the contrary, the variations in the proportions of total income possessed by stated proportions of the population are a matter of secondary importance.[49] According to him the most serious aspect of the problem is the extreme skewness of the income frequency curve, and the diminution of this skewness is a favorable symptom. From this point of view, the percentage P—which, as the above tables show, oscillates about 0.8—can be taken as a rough index of the asymmetry of the income curve in the present economic and juridical organization of society. Clearly P would be equal to 0.5 if the curve were symmetric.

If the structure of capitalistic society had evolved in accordance with Marx's forecast, the nineteenth century would have witnessed a steady increase of P, that is to say, of the relative number of people with incomes below the average income. But the available statistics conclusively show that this did not happen.

A FALLACIOUS INTERPRETATION OF INCOME STATISTICS

14. Pareto's device of plotting cumulated income data on double logarithmic paper has proved very useful in the analysis of income distributions. It has made possible the solving of a question concerning the correct interpretation of income statistics, which had been dis-

[48] P will always vary in the same sense as the index of inequality D only in the case of Pareto's first law, since in that case both P and D depend only on α.

Another characteristic feature of P is that, in a given distribution, it is the same for any portion of a given curve of incomes, when the whole curve follows Pareto's first or second law. (See the present writer's article, "On Pareto's Law," *Journal of the Royal Statistical Society*, 1937, which is a complement to the present paper.) Indeed, in both cases,

$$P = 1 - \left(\frac{\alpha - 1}{\alpha}\right)^{\alpha}.$$

[49] See article quoted above.

cussed for decades, more especially among German statisticians, without any positive result being reached. This discussion affords a striking example of the confusion and of the errors to which the use of loose methods could lead even prominent authors.

In their inquiries on the changes in the distribution of incomes, Engel and other older statisticians started from a given classification of incomes, which was kept fixed at two different dates, and compared the number of persons, who were found in each class interval at the two epochs. Engel's calculations show that the higher the income, the more rapid was the percentage increase in the number of income recipients between 1852 and 1873. This proved, in the author's opinion, that "the fact of the concentration of wealth could not be open to doubt."[50] Engel's conclusion was confirmed, some decades afterwards, by Wagner. In his inquiries on the distribution of personal incomes in Prussia during the nineteenth century,[51] Wagner also stated that the relative increase in the number of taxpayers had been much more considerable in the higher than in the lower income groups; and this led Wagner to conclude that "the difference between social classes, in so far as it is measured by the size of the income, has increased."[52] But other German statisticians—such as Böhmert, Soetbeer, and J. Wolf[53]—did not accept Engel's interpretation of income statistics. They contended that an improvement in the economic conditions of broad sections of the community must result in a certain number of income recipients shifting from the lower to the higher income brackets. Therefore, the fact that the rate of increase of the number of taxpayers is more rapid in the higher class intervals is the sign, not of an increasing concentration of income, but of a rapid rise of small and moderate incomes. The latter authors just quoted state that the inequality of incomes diminished in Prussia and Saxony during the

[50] *Op. cit.*, p. 142.

[51] "Zur Methodik der Statistik des Volkseinkommens und Volksvermögens," two papers published in *Zeitschrift des königl. preussischen statistischen Bureaus*, Berlin, 1904.

[52] The increase in the number of taxpayers in each class interval from 1878 to 1902 is shown by the following table (Wagner, *op. cit.*, p. 79).

Marks	Rate of increase	Marks	Rate of increase
900–2100	148.7 per cent	9600–28,800	270.8 per cent
2100–3000	109.3 per cent	28,800–96,000	332.4 per cent
3000–6000	140.6 per cent	above 96,000	635.5 per cent
6000–9600	206.2 per cent		

[53] Soetbeer, "Volkseinkommen im preussischen Staate, 1876 und 1888," *Jahrbücher für Nationalökonomie*, 1889; Böhmert, "Sächsische Einkommensteuerstatistik, 1875–1894," *Zeitschrift des sächsischen statistischen Bureaus*, 1895; J. Wolf, *Sozialismus und kapitalistische Wirtschaftsordnung*, 1892, p. 233.

nineteenth century. From the same data exactly the opposite conclusions were drawn!

15. Let AB in the accompanying double logarithmic chart, Figure 2, represent the income curve (cumulated data) on the assumption that it follows Pareto's first law, and let us suppose that, as happens most frequently, only the portion of the income curve between the two incomes Oh and Ox is known from published data. Let OM and ON indicate the logarithms of the numbers of persons with incomes above Oh and Ox respectively. Suppose now a proportionate addition to be made to each individual income, so that the degree of inequality is

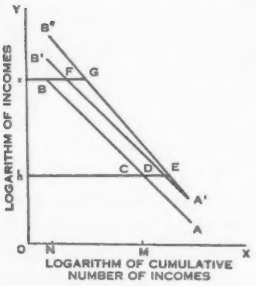

FIGURE 2.

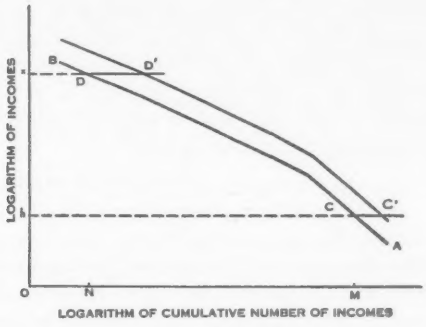

FIGURE 3.

unchanged. The income curve will shift, say, to the position of $A'B'$. Since the horizontal intercepts between the two lines AB and $A'B'$ measure the rate of increase of taxpayers, it will be seen from the chart that between the incomes Oh and Ox the rate of increase is the same for all groups of taxpayers ($CD = BF$).

Let us now suppose that more than proportionate additions are made to the higher incomes, so that the income curve takes the position of $A'B''$, indicating greater inequality. In this case, the percentage increase of the number of people with incomes above some stated limit will be the greater, the higher the income ($BG > CE$). Therefore, when the number of income receivers increases between two epochs more rapidly in the higher than in the lower incomes, this fact might well be interpreted as a sign of increased inequality of incomes, when the actual distribution follows Pareto's first law. (Of course, the influence of changes in population, of migratory movements, and so on, should also be taken into consideration.) In this case the objections against Engel's and Wagner's interpretation are not well founded.

But the conclusion is different in the case of Pareto's second law.

(See curve AB of Figure 3 which represents the distribution of properties in Prussia in 1911.)

In this case, when proportionate additions are made to each individual income, so that the degree of inequality does not vary, the percentage increase of the number of taxpayers with incomes above some stated limit will be the more rapid—as the horizontal intercepts between the two curves of Figure 3 show—the higher the income.[54] In this case, therefore Engel's and Wagner's interpretation might lead to quite fallacious inferences. However, as a matter of fact their conclusions were in the main sound, owing to the circumstance—of which, of course, both authors were unaware—that the distribution of Prussian incomes when plotted on a double logarithmic chart approaches to a straight line, as Figure 1 shows.

In the last few years several statisticians have again aroused some doubts as to the appropriateness of the fixed-income-classification method.[55] On this point it might be observed that the question has been already definitely solved.[56]

CONCLUDING REMARKS

16. Pareto's law is but an "empirical" and not a rational law of the distribution of incomes. Pareto did not attribute any importance to this distinction, which was stressed by Edgeworth in his controversy with Pareto. Afterwards some interesting attempts have been made (I refer more especially to Professor Vinci's work)[57] to give a rational

[54] In this example a is positive. When a is negative and proportionate additions are made to each individual income, the rate of increase in the number of taxpayers in each class interval will be the less rapid, the higher the income. A more detailed analysis will be found in the present writer's papers "Über die Methoden der Einkommenverteilungsstatistik," *Jahrbücher für Nationalökonomie*, 1907, and "Kritische Betrachtungen über einige Methoden der Einkommensstatistik," *Statistische Monatsschrift*, 1915.

It should be added that the index of inequality D is smaller for the curve CD (Figure 3) than for the curve $C'D'$ since in the latter a is greater than in the former curve, while α remains the same. It follows that the comparison of values of D referring to different epochs may also lead to fallacious conclusions, when the distribution does not follow Pareto's first law, and only a portion of the actual curve of incomes is available.

[55] See Lord Stamp's observations in the *Journal of the Royal Statistical Society*, 1937, No. 4, p. 668.

[56] See more especially Professor Winkler's article "Einkommen" in *Handwörterbuch der Staatswissenschaften*.

[57] Vinci, "Calcolo delle probabilità e distribuzione dei redditi nel pensiero di Pareto," *Giornale degli Economisti*, 1924 and *Manuale di statistica*, Vol. 2, p. 99; D'Addario, *Sulla curva dei redditi di Amoroso*, Bari, 1936, p. 54 and *Le trasformate euleriane*, Bari, 1936, Chapter 4. See also A. L. Bowley, "The Action of Economic Forces in Producing Frequency Distributions of Income, Prices, and Other Phenomena," ECONOMETRICA, Vol. 1, October, 1934, pp. 358–372.

explanation of the form of the income curve. It is assumed that some disturbing forces, of which Vinci gives the analytical expression, bring about a distortion of the fundamental curve, which is supposed to be the result of a large number of unbiased forces. There are obstacles in the way of the acquisition of an income, and personal circumstances favoring this acquisition: the latter include the differences in the initial position of the income receivers, which determine an unequal distribution of opportunities.

Pareto's equation of the income curve, even in its more developed form, sometimes fails to represent with sufficient approximation actual distributions. In any case Pareto's equation—which states that the number of income receivers diminishes when their income increases—cannot fit the portion of the income curve below the modal, or most common, income. Other equations have been proposed—by March, Vinci, Amoroso, D'Addario, and others[58]—which are often more in accordance with actual data than Pareto's curve. But, from a practical point of view, it is important to stress the fact—which is confirmed by many observations—that Pareto's curve, even when it diverges substantially from the actual curve, generally fits very well limited portions of it, with different values of α. It follows that Pareto's formulae can be used to good advantage for interpolation, or when—as is generally the case with prewar statistics—the aggregate income possessed by each group of income receivers is not known. Then, according to Pareto's first law, the aggregate income in the interval from x_1 to $x_2 (x_1 < x_2)$ is

$$\frac{\alpha}{\alpha - 1} (x_1 N_1 - x_2 N_2)$$

where N_1 and N_2 are the numbers of taxpayers with incomes above x_1 and x_2 respectively.[59]

[58] L. March, "Quelques exemples de distribution de salaires," *Journal de la Société de Statistique de Paris*, 1898; F. Vinci, "Nuovi contributi allo studio della distribuzione dei redditi," *Giornale degli Economisti*, 1921; L. Amoroso, "Ricerche intorno alla curva dei redditi," *Annali di matematica pura ed applicata*, 1924–25; R. D'Addario, *Sulla curva dei redditi di Amoroso*, and *Curve di concentrazione, elasticità, flessibilità, densità media e densità marginale dei redditi*, Bari, 1936.

[59] The following table—which refers to the distribution of income in Holland (1927–1928)—shows that the agreement between actual and calculated amounts of income might be very close, even when Pareto's Law does not well fit the whole curve, as is indicated in the present case by the fact that the value of α is not the same in each class interval.

17. But even if it be granted that several statements and inferences of Pareto cannot withstand criticism and the test of actual facts, this by no means diminishes the importance of Pareto's law. Pareto's conclusions as to the relation between the average income and the degree of inequality might not be accepted; yet he has drawn attention to a fundamental problem of economic policy. He overlooked the significance of the changes in the distribution of personal incomes from place to place and from time to time; yet recent statistics confirm that the income distribution, when intervals of sufficient length are taken, shows a striking stability. Pareto's measure of the inequality of incomes—the ratio of the number of individuals with incomes equal to, or greater than x, to the number of individuals with incomes smaller than x—has been generally abandoned; and Pareto's parameter α is not always a good index, since it assumes a definite law of distribution of income which sometimes is not in accordance with actual statistics. Other more appropriate indexes of inequality have been elaborated. Yet Pareto conceived for the first time the idea of measuring the inequality in the distribution of incomes, suggesting a method which, though objectionable, allows of making comparisons between different places or epochs. Moreover if in Pareto's definition x is conceived as that income which in each place or epoch, taking into account changes of prices, allows the minimum decent standard of living, Pareto's test—though inappropriate as a measure of inequality—becomes a very good index of economic welfare; since it is very important to know what proportion of the population fails to attain the minimum decent standard of living.[60] Provided this proportion diminishes substantially,

Class intervals (guilders)	Aggregate income (millions of guilders)		
	Actual	Calculated from Pareto's first law	Value of α
2,000– 3,000	646.3	664.1	2.08
3,000– 5,000	524.0	536.1	1.83
5,000– 10,000	415.0	422.6	1.65
10,000– 20,000	258.5	259.6	1.52
20,000– 30,000	113.3	113.5	1.53
30,000–100,000	225.9	225.0	1.59

The aggregate calculated income of all taxpayers is 2,221 million guilders, and differs but little from the actual aggregate income (2,183 million guilders). The value of α is calculated from Pareto's formula

$$\frac{N_2}{N_1} = \left(\frac{x_1}{x_2}\right)^{\alpha}.$$

[60] As to the questions aroused by the calculation of the "poverty line" see Bowley, *Wages and Income in the United Kingdom since 1860*, 1937, Chapter 4.

it might be a matter of little concern whether, at the same time, the small group of individuals at the top of the income scale gets a slightly larger percentage of total income than it got before. In Egypt, for instance, the distribution of incomes is probably less unequal than in England; yet, in view of the extreme poverty of broad sections of the Egyptian population, nobody would contend that social conditions are in Egypt more favorable than in England.

But the profound significance of Pareto's law lies in that it states the existence of an "order" in the distribution of individual incomes. Pareto's discovery is a positive contribution to the theory of the persistence of the social groups, which, in Lexis's conception, affords the basis for the explanation of the statistical regularities in human actions. Thanks to Pareto, another important field of social phenomena—besides those discovered by Graunt, Süssmilch, Quetelet, and Lexis—was shown to be dominated by statistical uniformities.

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L'OEUVRE ÉCONOMIQUE D'AUGUSTIN COURNOT*

Par René Roy

Augustin Cournot, qui naquit en 1801 à Gray, petite ville de Franche-Comte, se consacra d'abord aux sciences et particulièrement aux mathématiques. Admis à l'École Normale Supérieure, il obtint successivement ses diplômes de licencié ès sciences, licencié en droit, puis docteur ès sciences; sa thèse de doctorat le fit remarquer par Poisson qui lui ménagea une brillante carrière dans l'enseignement.

Outre des traductions d'ouvrages scientifiques et des essais concernant les mathématiques appliquées, Cournot s'orienta principalement vers la philosophie et l'économie politique; cette dernière branche de connaissances paraît d'ailleurs lui avoir fourni l'occasion d'appliquer à la matière sociale ses conceptions relatives aux lois du hasard, qu'il a développées en 1843 dans son *Exposé de la théorie des chances et des probabilités.*

Son oeuvre économique comprend trois ouvrages:

Recherches sur les principes mathématiques de la théorie des richesses (1838);

Principes de la théorie des richesses (1863);

Revue sommaire des doctrines économiques (1877).

Quant à son oeuvre philosophique, parmi laquelle se rencontrent fréquemment des passages traitant des phénomènes économiques, elle comprend essentiellement les ouvrages suivants, que nous mentionnerons à nouveau dans la suite de cet exposé:

Traité de l'enchainement des idées fondamentales dans les sciences et dans l'histoire (1861);

Considérations sur la marche des idées et des évènements dans les temps modernes (1872);

Matérialisme, Vitalisme, Rationalisme (1875).

L'OEUVRE ÉCONOMIQUE

Se proposant d'appliquer à la matière économique les méthodes qui avaient fait leurs preuves en d'autres domaines et pouvant être considéré à ce titre comme le fondateur de l'économie scientifique, Cournot expose entièrement ses concepts dans le traité de 1838 intitulé comme nous l'avons indiqué *Recherches sur les principes mathématiques de la théorie des richesses;* 25 ans plus tard, c'est-à-dire en 1863, parut un second recueil intitulé *Principes de la théorie des richesses*, qui se bornait

* Paper presented at the Fourth Annual Research Conference of the Cowles Commission for Research in Economics, Colorado Springs, July, 1938, in honor of the centenary of the publication of Cournot's *Researches into the Mathematical Principles of the Theory of Wealth.*

à développer les mêmes idées qu'auparavant, mais cette fois sans aucune intervention des symbôles mathématiques. Enfin, au déclin même de sa vie, Cournot consacra ses dernières forces à la rédaction de sa *Revue sommaire des doctrines économiques*, qui vit le jour en 1877. Ce serait toutefois restreindre l'oeuvre économique de cet auteur que de la limiter à ces seuls écrits, spécialement consacrés à la matière économique, car dans ses autres ouvrages et en particulier dans ceux où il développe ses conceptions philosophiques, comme le *Traité de l'enchaînement des idées fondamentales dans les sciences et dans l'histoire* (1861), les *Considérations sur la marche des idées et des évènements dans les temps modernes* (1872), *Matérialisme, Vitalisme, Rationalisme* (1875), nombreux sont les passages qui renferment de substantiels développements sur les phénomènes économiques, et sur la manière dont ils s'insèrent parmi les autres manifestations d'ordre social.

L'intérêt de Cournot, sa passion même pour les études économiques apparaissent avec plus de forces encore si l'on considère que les *Recherches* furent sa première oeuvre vraiment originale et que sa *Revue sommaire* parut en 1877, c'est-à-dire l'année même de sa mort. Cournot se tint d'ailleurs constamment au courant des choses traitant d'économie politique et son érudition dans ce domaine était immense: il avait lu Aristote, Montchrestien, les économistes anglais du XVIII[e] siècle, les physiocrates et en particulier Turgot, Adam Smith et les maîtres de l'école libérale, J.-B. Say, Ricardo, Stuart Mill, Mastia; il avait suivi de près l'évolution des problèmes monétaires et à l'étude de toutes ces questions, il avait apporté la pénétration de son esprit critique. D'une façon générale, Cournot fut toujours attiré vers les études sociales, ainsi que l'a observé le professeur Moore qui a minutieusement approfondi la vie de cet auteur et la nature de ses travaux.

Quant aux motifs qui peuvent être invoqués pour expliquer la prédilection de Cournot à l'égard des études économiques, il semble qu'on doive surtout retenir le fait que la matière sociale est singulièrement apte aux applications de la théorie des probabilités, comme en témoigne l'usage qui en a été fait à la démographie, à la théorie des assurances et d'une façon générale à cet ensemble de questions que Cournot désignait sous l'appellation "Arithmétique sociale."

Particulièrement familiarisé avec la pensée de Leibnitz qui, à raison même de sa culture universelle, avait entrevu déjà cette application du langage mathématique aux phénomènes sociaux, il n'est pas surprenant qu'à la lecture d'écrivains comme Ricardo, Cournot ait été naturellement porté à traiter ce genre de problème au moyen des mathématiques et à en généraliser l'emploi par un exposé synthétique de cet ordre de connaissances.

Dans la préface des *Recherches* il indique au reste fort clairement les

raisons pour lesquelles il recourt au symbôle mathématique:

> L'emploi des signes mathématiques est chose naturelle toutes les fois qu'il s'agit de discuter des relations entre des grandeurs, et lors même qu'ils ne seraient pas rigoureusement nécessaires, s'ils peuvent faciliter l'exposition, la rendre plus concise, mettre sur la voie des développements plus étendus, prévenir les écarts d'une vague argumentation, il serait peu philosophique de les rebuter, parce qu'ils ne sont pas également familiers à tous les lecteurs et que l'on s'en est quelquefois servi à faux.

Cette même préface du traité de 1838 permet aussi d'apprécier la manière dont l'auteur fut incité à utiliser le langage mathématique pour la présentation des phénomènes économiques, à la suite des réflexions que lui suggéra la lecture des auteurs classiques:

> Il y a des auteurs tels que Smith et Say, qui ont écrit sur l'économie politique en conservant à leur style tous les agréments de la forme purement littéraire; il y en a d'autres, comme Ricardo, qui, abordant des questions plus abstraites ou recherchant une plus grande précision, n'ont pu éviter l'algèbre, et n'ont fait que la déguiser sous des calculs arithmétiques d'une prolixité fatigante. . . .

Il définit ainsi clairement son but:

> Je me propose d'établir dans cet essai que la solution des questions générales auxquelles donne lieu la théorie des richesses, dépend essentiellement non pas de l'algèbre élémentaire, mais de cette branche de l'analyse qui a pour objet des fonctions arbitraires assujetties seulement à satisfaire à certaines conditions.

S'il n'est pas le premier qui songe à utiliser le langage mathématique pour l'analyse des phénomènes économiques, puisque déjà l'Ingénieur des Ponts et Chaussées Isnard s'y était appliqué en 1771, puis Von Thünen en 1826, Cournot est bien le premier qui recourt systématiquement à la présentation mathématique pour constituer l'esquisse d'une théorie cohérente des phénomènes économiques envisagés dans toutes leurs généralités. D'autre part, l'auteur compte faire appel pour la présentation de ses exposés, non pas à l'algèbre élémentaire, mais à l'analyse mathématique, c'est-à-dire à des connaissances d'un niveau relativement élevé; il n'est d'ailleurs question pour le moment que d'utiliser des fonctions indéterminées, ce qui marque bien l'intention de se limiter aux recherches d'ordre qualitatif; nous verrons néanmoins que Cournot ne se refuse pas à concevoir la possibilité de pousser les investigations jusqu'au stade quantitatif, et qu'il a ainsi montré la voie aux adeptes contemporains de l'Économétrie, qui s'efforcent précisément d'aboutir à la détermination d'éléments numériques.

Les *Principes de la théorie des richesses*, publiés en 1863, n'ont été dictés à Cournot que par l'insuccès du premier traité, qui, selon l'auteur, avait rebuté le lecteur par l'usage des symbôles mathématiques; aussi, le second ouvrage est-il délibérément dépouillé de tout appareil mathématique; il ne renferme au reste rien d'essentielle-

ment nouveau par rapport au premier essai, qui reste l'oeuvre maîtresse d'Augustin Cournot dans le domaine économique. Enfin, la *Revue sommaire des doctrines économiques*, si elle n'a pas été dictée par ce besoin de polémique dont un esprit supérieur parvient toujours à s'affranchir, a été cependant rédigée en partie pour confondre ses déctracteurs qui lui reprochaient de méconnaître l'oeuvre économique de ses devanciers.

L'essai de 1838 contenant l'essentiel des recherches économiques de Cournot, et cet ouvrage ayant véritablement servi de base aux travaux ultérieurs de l'école mathématique, il suffit d'exposer les passages vraiment caractéristiques de cette oeuvre pour démontrer l'importance de la contribution apportée par l'auteur à l'édification de l'économie scientifique.

DE LA LOI DU DEBIT

En abordant l'étude de la demande ou de la "loi du débit," qui sert de base à tous ses développements ultérieurs, Cournot prend soin d'exposer le principe auquel il entend se référer:

> Nous n'invoquerons qu'un seul axiome, ou, si l'on veut, nous n'emploierons qu'une seule hypothèse, savoir que chacun cherche à tirer de sa chose ou de son travail la plus grande valeur possible.

C'est en substance le principe édonistique auquel ont recouru depuis lors les maîtres classiques de l'économie mathématique, Walras, Pareto, principe qu'ils ont mis en oeuvre, soit au moyen des courbes ou fonctions d'utilité, soit au moyen de l'ophélimité.

Cela posé, Cournot définit ainsi qu'il suit sa loi de la demande ou du débit:

> Admettons donc que le débit ou la demande annuelle D est, pour chaque denrée, une fonction particulière $F(p)$ du prix p de cette denrée. Connaître la forme de cette fonction, ce serait connaître ce que nous appelons la loi de la demande ou du débit.

Après avoir déterminé les propriétés essentielles de cette fonction $F(p)$ (continuité, décroissance), l'auteur envisage son utilisation en se conformant au principe général qu'il a posé dans la préface et sur lequel nous avons eu l'occasion d'insister:

> . . . il ne serait pas moins à propos d'introduire, au moyen d'un signe indéterminé, la loi inconnue de la demande, dans les combinaisons analytiques: car on sait que l'une des fonctions les plus importantes de l'analyse consiste précisément à assigner des relations déterminées entre des quantités dont les valeurs numériques et même les formes algébriques sont absolument inassignables.

Cournot, tout en marquant l'intérêt d'une telle conception qualitative, dont il démontre d'ailleurs la fécondité dans la suite de son exposé, anticipe néanmoins sur les travaux de ses successeurs immédiats et

trace le cadre dont s'inspirent actuellement les adeptes de l'économétrie c'est-à-dire ceux qui s'efforcent d'atteindre le mesurable:

> Puisque tant de causes morales, et qu'on ne peut énumérer ni mesurer, influent sur la loi de la demande, il est clair qu'on ne peut pas s'attendre à ce que cette loi puisse être exprimée par une formule algébrique, pas plus que la loi de mortalité et que toutes celles dont la détermination rentre dans le domaine de la statistique, de ce qu'on appelle également l'arithmétique sociale. Ce serait donc à l'observation à fournir les moyens de dresser entre des limites convenables une table des valeurs correspondantes de D et de p; après quoi l'on construirait, par les méthodes connues d'interpolation ou par les procédés graphiques, une formule empirique ou une courbe propres à représenter la fonction dont il s'agit; et l'on pourrait pousser la solution des problèmes jusqu'aux applications numériques.

Si l'on songe au caractère particulièrement hardi de ces conceptions à l'époque où elles furent présentées, on s'expliquera facilement les motifs de leur insuccès; sans être en mesure d'aborder les observations qui le conduiraient à la détermination d'éléments quantitatifs, Cournot n'en donne pas moins de très substantielles indications sous les divers aspects de la loi du débit selon la nature des denrées que l'on considère:

> La demande pourrait être en raison inverse du prix; ordinairement, elle croit ou décroit dans une proportion beaucoup plus rapide: observation qui s'applique surtout au plus grand nombre des produits manufacturés. D'autres fois au contraire, la variation de la demande est moins rapide; ce qui paraît (chose singulière) s'appliquer également et aux choses les plus nécessaires, et aux choses les plus superflues. Le prix des violons, celui des lunettes astronomiques baisserait de moitié, que probablement la demande ne doublerait pas; . . . Le bois de chauffage, qui est au contraire une denrée des plus utiles, pourrait doubler de prix, par suite des progrès du défrichement ou de l'accroissement de la population, probablement bien avant que la consommation annuelle de bois eût été réduite de moitié; un grand nombre de consommateurs étant disposés à retrancher sur les autres dépenses plutôt que de se passer de bois.

Certes, ce sont là des considérations qui nous paraissent un peu sommaire depuis que Marshall nous a mis en possession d'une donnée précise, le coefficient d'élasticité, depuis surtout que les recherches des économètres ont permis de procéder à la détermination de ces coefficients par voie d'observation statistique; notons malgré tout que Cournot, en considérant la loi de variation du débit en raison inverse du prix, a tout de suite décelé l'importance d'un tel critèrium pour les distinctions à établir en matière de demande. Il n'a d'ailleurs pas manqué d'y insister, en abordant l'étude de la valeur globale d'une denrée.

Ayant introduit cette notion de valeur globale, produit du prix p de la denrée par le débit $F(p)$ correspondant à ce prix, Cournot, au moyen d'un raisonnement purement mathématique, fondé sur la notion de continuité, ainsi que sur l'existence d'un prix limite-fini et d'un débit-limite également fini, démontre l'existence d'une valeur optima du prix

répondant à un maximum de la valeur globale; mais loin de se contenter d'une conclusion théorique, il en révèle toute la fécondité dans le domaine des problèmes concrets:

> La construction d'une table où l'on introduirait ces valeurs serait le travail le plus propre à préparer la solution pratique et rigoureuse des questions qui se rattachent à la théorie des richesses.

Il développe même son exposé jusqu'à la description sommaire de la tâche qui, en cette matière, incombera aux statisticiens:

> On devra donc demander d'abord à la statistique commerciale de distribuer les denrées d'une haute importance économique en deux catégories, selon que leurs prix courants restent inférieurs ou supérieurs à la valeur productrice du maximum du produit $pF(p)$. Nous verrons que beaucoup de problèmes économiques comportent des solutions différentes, selon que la denrée dont il s'agit appartient à l'une ou à l'autre de ces deux catégories.

N'est-ce point là poser d'une manière très précise la distinction aujourd'hui classique entre les produits à "demande élastique" et les produits à "demande inélastique"? Sur ce point particulier, la pénétration de Cournot s'exerce avec un rare bonheur et sans doute y voyons-nous un éclatante justification pour son titre de précurseur.

De cette brève analyse portant sur la "loi du débit," retenons en définitive la solidité de l'oeuvre qui a servi de support à toutes les études ultérieures sur les courbes de demande. Certes, Cournot, en ne faisant intervenir que le prix de la seule denrée considérée, se montre en désaccord avec les fondateurs de l'équilibre économique qui présentent au contraire le système des prix comme un ensemble dont tous les éléments, liés par des rapports de mutuelle dépendance, sont soumis à des variations concomitantes. Ce serait pourtant une erreur d'imaginer que Cournot s'est mépris sur la portée de ses recherches: sa formation mathématique l'avait trop familiarisé avec la notion des ensembles, pour qu'il fût tenté de ne pas regarder sa loi du débit comme une simple approximation; Walras, puis Pareto, tout en faisant intervenir des fonctions à plusieurs variables, ont cependant reconnu le parti que l'on pouvait tirer de la conception à laquelle s'était référé Cournot pour l'exposé de sa loi du débit, et depuis lors, nos idées ne se sont guère modifiées à cet égard.

DU MONOPOLE

L'importance qu'attribuait Cournot à la théorie du monopole se trouve pleinement exprimée dans la fin du chapitre concernant la loi du débit:

> On doit, dans toute exposition, procéder du simple au composé: l'hypothèse la plus simple, quand on se propose de rechercher d'après quelles lois les prix s'établissent, est celle du monopole, en prenant ce mot dans le sens le plus

absolu, ce qui suppose que la production de la denrée est dans une seule main . . . et d'ailleurs, après que nous l'aurons étudiée, nous pourrons analyser avec plus de précision les effets de la concurrence des producteurs.

S'il convient de faire les plus expresses réserves en ce qui touche un tel processus qui consiste à utiliser la théorie du monopole comme support de la détermination des prix, en régime de concurrence, limitée ou indéfinie, il importe en revanche d'insister sur les acquisitions précieuses qui ont été réalisées par Cournot dans le seul domaine du monopole, car ces acquisitions, enrichies et complétées depuis lors par Dupuit, servent encore de base aujourd'hui à toutes les applications concrètes en matière de monopole.

Bien entendu, Cournot s'empresse d'utiliser les résultats qu'il a déjà obtenus en traitant du débit, et il se réfère tout spécialement aux résultats concernant la valeur globale qui, dans le cas du monopole, devient la recette brute et doit être rapprochée des frais d'exploitation, pour faire apparaître le bénéfice du monopoleur, que celui-ci s'efforce de rendre aussi grand que possible.

Pour la recherche de cet optimum, qui permet d'obtenir ce qu'on appelle depuis lors le "prix de monopole," deux cas sont distingués: celui où les frais de production peuvent être considérés comme nuls ou indépendants de la quantité produite et celui où les frais de production doivent être au contraire, tenus pour variables en fonction de la quantité produite. L'examen de ce deuxième cas permet d'ailleurs à notre auteur d'introduire dans la discussion la notion du "coût de production marginale," c'est-à-dire du prix de revient de la dernière unité produite, qui sans être explicitement désigné sous ce vocable précis, est mis en lumière d'une façon très claire. Considérant en effet l'ensemble des frais de production comme une fonction de la demande D, soit $\phi(D)$ Cournot observe que le dérivée $\phi'(D)$ peut obéir à des lois de variation qui diffèrent selon la nature des marchandises:

La fonction $\phi'(D)$ est, selon la nature des forces productrices et des données pr[illegible]ites, susceptible de croitre ou de décroitre quand D augmente.

Ainsi apparaissent déjà les deux familles de production qu'envisagent aujourd'hui tous les traités: celle des produits à prix de revient croissants et celle des produits à prix de revient décroissants. En ce qui concerne les agents naturels, Cournot précise en ces termes les conséquences de la loi du rendement décroissant:

Quand il s'agit de l'exploitation des terres arables, des mines, des carrières, de la richesse éminemment foncière, la fonction $\phi'(D)$ est croissante avec D.

Ainsi que nous l'avons fait observer, c'est aux principes dégagés par Cournot, puis développés par Dupuit, que recourent encore à l'heure actuelle, plus ou moins consciemment, ceux qui se proposent d'accroître

le rendement des services monopolisés par la mise en oeuvre d'une tarification commerciale appropriée; à ce titre, l'oeuvre de Cournot et de ses continuateurs immédiats conserve un éminent caractère d'actualité.

Beaucoup plus discutable en revanche est le processus par lequel Cournot, utilisant les résultats acquis dans la théorie du monopole, examine les corrections à faire subir à ces résultats et en particulier à la détermination du prix de vente, lorsqu'on se trouve en présence de plusieurs producteurs, en nombre limité d'abord, puis illimité. Actuellement au contraire, c'est la concurrence indéfinie qui se présente comme l'hypothèse normale, tandis que le monopole fait l'objet d'une étude complètement distincte. A notre sens, la position prise par Cournot ne peut guère se justifier que par la formation abstraite de son esprit, et par la nature toute géométrique de son raisonnement: pour un tel esprit, le monopole représente évidemment une hypothèse nettement définie, qui permet d'aboutir à des conclusions précises en recourant à ces fonctions indéterminées que nous avons vu apparaître dès la préface des *Recherches*. Partant de cette hypothèse simple, Cournot aborde ensuite le cas plus délicat de la concurrence limitée et termine par celui de la concurrence indéfinie. C'est là, semble-t-il, faire trop bon marché de la réalité qui, surtout il y a un siècle, laissait apparaître le monopole ou la concurrence limitée comme des situations exceptionnelles, relativement à la concurrence indéfinie qui représentait le cas général.

Du point de vue théorique cependant l'étude de la concurrence limitée a mis Cournot sur la voie d'un problème qu'il a résolu en supposant que chaque producteur se comportait comme s'il exploitait un monopole; la validité de cette hypothèse a suscité de nos jours des controverses animées qui, sans aboutir à des conclusions définitives, ont toutefois permis de préciser grandement nos idées sur ce problème du "polypôle" qui avait déjà retenu l'attention du mathématicien Joseph Bertrand, et qui, à raison même de l'évolution relative aux conditions de la production moderne, est loin de rester dépourvu d'intérêt lorsqu'on se propose d'aborder rationnellement l'étude des questions concrètes.

DU CHANGE

Le chapitre que Cournot a consacré dans son essai de 1838 aux phénomènes de change occupe à notre avis une place tout à fait à part, car plus encore que la loi du débit, ce bref exposé conserve après un siècle toute sa valeur, et l'on peut même dire qu'à cet égard, il ne s'est produit aucun changement notable dans nos conceptions.

Se plaçant exclusivement dans l'hypothèse de la circulation métallique, Cournot la définit d'une manière très précise:

Nous supposerons donc que tous les peuples commerçants aient adopté la même unité monétaire, par exemple, le gramme d'argent fin, ou, ce qui revient au même, que le rapport de chaque unité monétaire au gramme d'argent fin soit invariablement fixé.

Puis, envisageant chacun des termes qui figurent dans la balance des comptes de deux pays, il montre comment le solde de cette balance détermine de petites variations de change qui n'entrainent de transferts de métal qu'au délà d'une certaine limite:

Toutes les fois donc que le rapport $m_{2,1}/m_{1,2}$ ne différera

de l'unité que d'une quantité moindre que le prix du transfert de l'unité monétaire d'une place à l'autre, le compte se soldera entre les deux places sans transfert réel d'argent, et par le seul effet du cours du change.

N'est-ce point formuler en termes parfaitement clairs et concis la théorie que l'on a depuis lors appelée "théorie des points d'or," en raison de l'utilisation presque universelle de l'or comme étalon monétaire?

Cournot passe ensuite à la description de l'arbitrage pour en déduire le système de relations qui lient les cours des changes d'un ensemble de pays considérés deux à deux:

Ainsi l'on a généralement $C_{i,k} := C_{i,1}.C_{1,k}$: ou du moins, si cette relation cesse momentanément d'être satisfaite, les négociations de banque tendent sans cesse à la rétablir.

Outre l'intérêt que présente cet exposé pour l'analyse de phénomènes monétaires, il offre à notre avis l'intérêt plus grand encore de constituer le premier exemple d'une synthèse qui permet de mettre clairement en évidence la nature et le nombre des variables considérées dans le système économique, ainsi que les relations qui existent entre elles. En d'autres termes, Cournot montre ainsi la voie aux fondateurs de l'équilibre économique, qui s'efforceront précisément de dégager le degré de liberté des systèmes économiques, en confrontant le nombre des variables avec le nombre des relations qui les lient. Or, c'est là un problème capital dans toute étude rationnelle des phénomènes économiques.

Le chapitre de Cournot sur le change qui, par sa rigueur toute géométrique, évoque les développements de l'analyse combinatoire, ne saurait être cependant regardé comme un édifice purement abstrait, car il s'apparente de très près à la réalité des phénomènes concrets; on conçoit d'ailleurs qu'il en soit ainsi, puisque les phénomènes monétaires, par leur sensibilité même, et par la nature du cadre général dans lequel ils se développent, devaient particulièrement convenir à une présentation de forme analytique.

Le caractère concret des résultats acquis est au reste mis en lumière par le conseil que donne Cournot à propos du bi-métallisme:

Ce sera en vain que les Gouvernements, constituant leurs systèmes monétaires, fixeront un rapport légal entre la valeur de l'or et celle de l'argent (comme en France, où la loi assigne à ce rapport la valeur 15,5).

Nous sommes en mesure de constater par cet exemple combien Cournot s'attachait dans certaines parties de son oeuvre économique à suivre le détail des phénomènes concrets; tandis que dans l'étude de la demande ou du monopole, il se présente à nous comme un penseur abstrait, non dépourvu certes de qualités d'observation, mais désireux avant tout de construire un édifice logique, l'étude du change nous offre au contraire l'exemple d'une matière qu'il a traitée en homme familiarisé avec la technique de l'arbitrage et avec tous les procédés auxquels ont recouru les divers États pour l'édification de leurs systèmes monétaires.

L'INFLUENCE DE COURNOT DANS LE DOMAINE ÉCONOMIQUE

De même que pour ses travaux de philosophie et d'histoire, l'oeuvre économique de Cournot connut peu de succès au cours du XIXème siècle, surtout en France. Nous pensons en trouver une preuve dans le silence manifesté à cet égard par un autre économiste français, l'Ingénieur des Ponts et Chaussées Jean Dupuit, qui fut le contemporain de Cournot.

Le premiere mémoire publié par Jean Dupuit en 1844 dans les *Annales des Ponts et Chaussées*, est consacré à l'utilité des travaux publics et d'une façon plus générale à la définition et à la mesure de l'utilité des biens économiques; l'auteur, précisant la distinction déjà établie par Adam Smith entre la "valeur d'échange" et la "valeur d'usage," recourt précisément à l'écart de ces deux grandeurs pour mesurer l'utilité d'une richesse; il se trouve ainsi conduit à envisager les courbes de demande, qui traduisent les variations du débit en fonction de la valeur d'usage attribuée à la marchandise considérée par les diverses couches d'acheteurs. Or, s'il est frappant d'observer la concordance des idées émises par les deux auteurs français tant sur cette question des courbes de demande que sur la théorie du monopole, il est plus significatif encore de noter que Dupuit ne fait aucune allusion aux *Recherches* de 1838 et semble ainsi complètement ignorer l'oeuvre d'Augustin Cournot. Certes, Dupuit ne s'est pas placé au même point de vue: tandis que Cournot fut surtout guidé par son goût pour les généralisations d'ordre philosophique et par ses conceptions touchant à la notion de probabilités, Dupuit se présente à nous comme un praticien qui domine les problèmes techniques à la solution desquels ses obligations professionnelles l'ont contraint de refléchir; il est toutefois intéressant de noter que 6 ans après la publication des *Recherches*, un penseur comme Dupuit, non dépourvu d'érudition, ne songe même pas

à citer cette oeuvre vraiment capitale pour l'utilisation des mathématiques à l'étude des phénomènes économiques.

Cependant, l'oeuvre économique de Cournot faisait lentement son chemin et suscitait à l'étranger, en Italie, en Angleterre, et aux États-Unis, un intérêt confinant souvent à l'admiration; depuis lors, l'influence de Cournot n'a fait que grandir, et malgré l'édification de l'équilibre économique, par l'École de Lausanne, il semble qu'il y ait encore place pour les conceptions plus modestes de Cournot et de Dupuit, et que pour certaines applications de la statistique ou des mathématiques aux phénomènes économiques, les hypothèses simplificatrices de ces deux auteurs conservent encore leur intérêt.

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THE DUPUIT TAXATION THEOREM

By Ragnar Frisch

On reading Professor Hotelling's interesting paper "The General Welfare . . . " which appeared in the July, 1938 issue of Econometrica, it occurred to me that a word of comment may be useful in order to prevent that more be read into the fundamental theorem of the paper and its conclusions than they really contain.

The generalized Dupuit theorem proved by Professor Hotelling does *not* state that satisfaction is necessarily *reduced* if we increase the portion of the total tax levy which is represented by excise taxes and correspondingly reduce the portion represented by an income tax, and that satisfaction is *increased* by a change in the opposite direction. In other words the theorem does not state that total satisfaction is necessarily reduced if the total tax burden (supposed given) is shifted in the direction *away from* income taxes and increased if the burden is shifted *towards* income taxes. Professor Hotelling's statement, "The change from income to excise taxes has resulted in a net loss of satisfactions. Conversely, . . . " (page 251) may suggest such an interpretation, although of course Professor Hotelling did not intend to convey this idea. The fact that the *direction* of change is immaterial is seen simply by noticing that Professor Hotelling's proof of the reduction in satisfaction is entirely independent of whether his magnitude δm (the change in the income tax) is positive or negative.

The only relevant question is whether the excise taxes are *proportional* or *nonproportional* to the prices that existed before the imposition of the excise taxes. It is the nonproportionality of the excise taxes, and only this, that produces a reduction in satisfaction. This reduction has nothing to do with the fact that excise taxes exist, nor even with the fact that they are high (positive or negative). Professor Hotelling's statement about "The inefficiency of an economic system in which there are excise taxes or bounties" (page 253) is certainly apt to be misunderstood.

Also in regard to the assumptions back of the analysis there are one or two points that ought to be brought out a little more explicitly. They necessitate rather important qualifications in the practical conclusions. I therefore believe it is worth while to go over the ground in another way which will bring out these points more clearly.

* * *

In doing this I shall adopt the usual index-number notation and terminology. Let p_0', p_0'', $\cdots$, and q_0', q_0'', $\cdots$, be prices and quantities in some base situation designated 0, where no taxes exist. The

total income of the individual (disregarding saving) in this situation is $\sum p_0 q_0$. Suppose that (positive, negative, or zero) excise taxes and a (positive, negative, or zero) income tax are imposed. All these taxes are assumed to be *special taxes* for the individual in question. The prices to him can therefore be assumed to change with the exact amount of the excise taxes.[1] Another way to express this fact is to say that the *underlying* price situation (the "market" prices) is unchanged while the government levies from this individual a certain amount for each unit of the various commodities he consumes (or produces).

If $p', p'', \cdots$, and $q', q'', \cdots$, are the prices and quantities in the new situation that emerges after the imposition of the taxes (no other changes having occurred), the individual excise taxes are consequently $p'-p_0', p''-p_0'', \cdots$, and the total amount levied in excise taxes is $\sum(p-p_0)q$. On the other hand the income tax imposed is obviously the difference between the income (spent) in the old situation and in the new, i.e., $\sum p_0 q_0 - \sum pq$. The total levy t (= "tax") from this individual (positive, negative, or zero) is consequently

$$\sum(p-p_0)q+\sum p_0 q_0-\sum pq=t. \tag{1}$$

Hence

$$\sum p_0 q=\sum p_0 q_0-t. \tag{2}$$

The last equation holds good for any q, no matter how the excise taxes are fixed, that is to say no matter what the new price situation p is. Consequently: *Proposition I.* If the total levy on this individual (income and excise taxes taken together) is t (given), the new quantity combination q must be one of those which could have been bought at the *original* prices for a sum which is equal to the *original* income minus the total levy t. And if nothing more than t is given, q may be any of these combinations. Geometrically expressed: The possibility region P of the new quantity point q is simply a hyperplane parallel to the original budget plane and at a distance (measured in income) equal to t. See Figure 1.

Let M be the point of the plane P where the highest level of satisfaction is reached (we may disregard as highly improbable the case where more than one point with this property exists). Then by definition no taxation system, and no organization of the other factors that determine the response of the individual, can—if the total tax t is given—be better than one which will induce him to move to M.

There are several systems that will lead to M. One consists in *telling* the individual that under any circumstances his income tax will be so adjusted to the other facts of the situation that his total tax

[1] See also footnote 8 in Professor Hotelling's paper.

will equal t. In this case he will move to M, no matter what excise taxes (positive, negative, or zero) are fixed, because he will then realize that P is his possibility plane. In discussing further alternatives let us suppose that the individual behaves as a "quantity adapter," that is, he acts as if the prices were *data* to him while he may freely choose his quantities. (This strategic type was the only one considered in Professor Hotelling's paper.) Then, a system leading to M is to levy no excise taxes whatsoever (so that the new prices will be identical

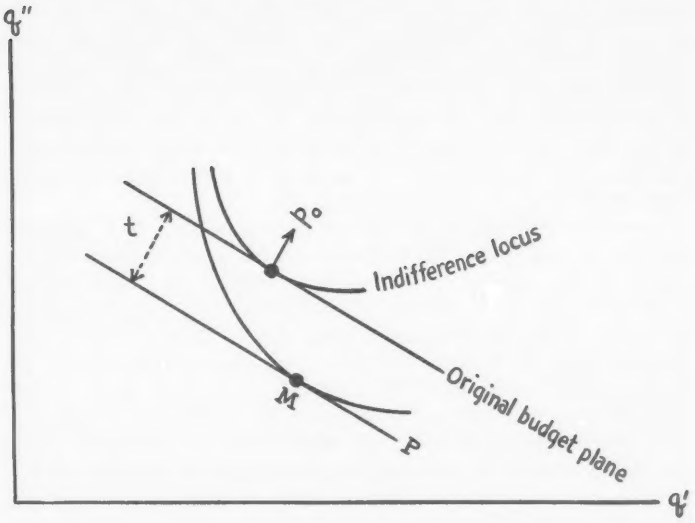

FIGURE 1

with the old). In this case the whole of t must, of course, be levied in the form of a fixed income tax. It is easy to see that this must lead to M. Indeed, P will now be the budget plane of an individual who behaves as a quantity adapter under the given price system p_0. This individual will, of course, choose M. Therefore: *Proposition II*. For a quantity adapter no tax system (with a given total levy) can be better than the one with no excise taxes.

Consider in particular the case $t=0$. In this case we may say that the measures taken in regard to this individual are *self-financed*. From II follows immediately: *Proposition III*. It is impossible to increase the satisfaction of a quantity adapter by means of a tax system which is self-financed with respect to him.

The system mentioned in II is only an example. There exists an

infinity of others that are equally good, namely all those whose excise taxes are *proportional* to the original prices $p_0', p_0'', \ldots$. Indeed, let the excise taxes be

$$\lambda p_0', \lambda p_0'', \cdots, \tag{3}$$

where λ is any proportionality factor, and let the income tax be fixed at

$$(1+\lambda)t - \lambda \sum p_0 q_0. \tag{4}$$

The budget plane of the new situation (the individual being a quantity adapter) is $\sum(1+\lambda)p_0 q = \sum p_0 q_0 - (1+\lambda)t + \lambda \sum p_0 q_0$, which obvicusly reduces to (2). Since λ is an arbitrary parameter there exist systems of this sort with excise taxes *as high as we please*. And all of them will lead to the optimum point M. In particular (4) shows that a pure income tax of t can—with the same revenue to the government and the same satisfaction to the individual—be replaced by a pure system of excise taxes of the form (3) with $\lambda = t/\sum p_0 q_0 - t$.

Inversely: *Proposition IV*. If the individual acts as a quantity adapter, and we assume the indicator to have a continuously turning normal in the vicinity of M[2] (and we disregard the case of several optimum points on P), the tax systems defined by (3) and (4) are the only ones (with *fixed* income and excise taxes) which will lead to the optimum point M. Indeed, if we have a tax system leading to M, the corresponding budget plane P' must pass through M. If its normal forms an angle with that of P at M, it will be possible to find points on P' in the vicinity of M where the satisfaction is greater than in M, but this is contrary to the fact that no tax system with the given t can give a larger satisfaction than the satisfaction in M. Consequently the budget plane P' corresponding to the tax system considered must be the plane P itself. Hence the excise taxes are of the form (3). But then the income tax must be $t - \sum \lambda p_0 q_M$ where q_M is the quantity combination in M. Since q_M lies on (2), $\sum p_0 q_M = \sum p_0 q_0 - t$, so that the income tax must be $t - \lambda(\sum p_0 q_0 - t)$ which is (4). This shows that any tax system where the excise taxes are not proportional to the original prices must lead to a decrease in satisfaction.

All the tax systems (3)–(4) that lead to the optimum point are disregarded by Professor Hotelling as "highly improbable," except the system with all the excise taxes equal to zero. And he consequently concludes about "The inefficiency of an economic system in which there are excise taxes or bounties" (page 253). I don't think that this is quite satisfactory. If we adopt the probability viewpoint, we must rather say that the case which Professor Hotelling chooses to consider,

[2] The proposition can no doubt be proved under less restrictive assumptions.

is *still more improbable*. It corresponds indeed to $\lambda = 0$, and this single value of λ is a set of measure zero as compared with the set of all values of λ. At any rate, the case which he chooses to consider is in my opinion too special to justify the far-reaching practical conclusions he draws.

The above analysis depends in a peculiar way on the *base* price situation from which we start. To exhibit the role played by the base situation it suffices to consider self-financed systems and the response of a quantity adapter to such systems. A system with given income and excise taxes can be visualized as a list of figures which the government sends the individual. Suppose that we start from the situation 0 and the government sends a list containing excise taxes nonproportional to the p_0 prices. This will by virtue of IV push the individual to a new quantity combination q_1 which produces less satisfaction than the original situation q_0. Suppose that the government now realizes this and wants to abolish the taxes. Suppose it does so by sending the individual—who now is in the point q_1—a second list which has exactly the same figures as the first list only with opposite signs, so that the sum of the two lists expresses a complete abolition of the tax system. What happens? If the individual considers the new list as a system of taxes to be applied by him to the q_1 situation, he will not move back to q_0. The new excise taxes are indeed nonproportional to the new prices and will consequently push him to some point q_2 yielding a still smaller satisfaction. If the government tried a series of "corrections," possibly with different tentative tax systems, each "correction" would only make things worse—or at best maintain the previous situation. Satisfaction would—just like entropy—move in one direction, whatever the government tried to do.[3]

In fact, this will of course *not* be the course of affairs. When receiving the second list the individual will *not* consider this list separately as a new tax system to be applied to his q_1 situation. He will consider the two lists together as one single system to be applied to the *original* situation q_0 and consequently he will move back from q_1 to q_0 upon receipt of the second list. This he will do provided q_0 *itself* had not been influenced by some still earlier tax system. If it had, he would have continued his search until he had reduced all the data to some "archaic" system $p_* q_*$ which in some absolute sense of the word is

[3] This fact is essentially the same as the one which in index-number theory is expressed by the upper and lower Konüs limits (see, for instance, my ECONOMETRICA "Survey," Vol. 4, January 1936, p. 25). Indeed the second quantity combination q_1 is *budgetary* with the first q_0, i.e., could have been bought for the 0 income at 0 prices, the third q_2 is budgetary with the second q_1, and so on. This sort of relationship is *not symmetric*. While q_1 is budgetary with q_0, q_0 is not budgetary with q_1. If they had been mutually budgetary (and the optimum point in the budget plane unique), q_1 and q_0 must have been identical.

"given" and independent of all government action. It is only when seen in relation to such a situation that the reaction of the individual will conform to propositions I–IV. In a modern society it would seem difficult to attach a tangible meaning to this concept of an absolute base situation. At any rate it would undoubtedly be of larger practical value if we could find propositions that need not be applied to the government measures *in toto*, but could be applied to separate measures or groups of measures.

All the above conclusions are invariant for an arbitrary (monotonically increasing) transformation of the indicator ϕ. The part of Professor Hotelling's argument that treats these and similar facts is also invariant in the same sense. But it should be noticed that another (and considerable) part of his argument does not have this property. One example is his argument (page 259) for sales at marginal cost (overhead to be covered by income and inheritance taxes) *even though* some classes may lose by such a system. I am not arguing against such a scheme, but I am saying that the arguments—pro and con—in this case depend on comparisons that are *not* invariant for an arbitrary (monotonically increasing) transformation of the indicator of a given individual, and that therefore the conclusions here presented do *not* follow from the mathematical proposition proved.

Finally I should like to add a word also on the Dupuit theorem in its original form, which has reference to the case of an excise tax on a single commodity. This is illustrated by Figure 1 in Professor Hotelling's paper (page 243). The shaded area will, it is stated, represent a "net social loss" caused by the taxation. The nature of this "net social loss" should be clearly understood. It emerges only because the money taken away from the consumers and producers is valued according to a "utility" principle—the consumers' and producers' surpluses—here expressed by the *integrals* under the curves $f(q)$ and $g(q)$ (notation of Professor Hotelling's paper), while the same money in the hands of the government is only valued according to its number of monetary units. The relevant question is, of course, what the government *does* with the money, and this must in all fairness be valued by applying the *same* sort of "utility" principle to the good which the money in question enables the government to produce and distribute to the consumers free of charge (or at reduced cost). If this is done, there will not be any "net social loss," but possibly a great gain, the size of the gain depending on the nature of the demand curve for the good produced by the government. The same sort of objection applies, I think, to quite a number of similar reasonings currently applied in taxation theory.

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THE RELATION OF PRICES TO MARGINAL COSTS IN AN OPTIMUM SYSTEM

By Harold Hotelling

In the July issue of Econometrica[1] I gave a new proof, taking account of the interrelations of commodities by methods not available in the times of Dupuit and Marshall, that in a specified sense maximum welfare requires that the quantity of each good consumed or produced by an individual shall be that corresponding to all sales being at marginal cost. This proposition has revolutionary implications, for example in electric-power and railway economics, in showing that society would do well to cut rates drastically and replace the revenue thus lost by subsidies derived largely from income and inheritance taxes and the site value of land.

Professor Frisch[2] correctly points out that these optimum quantities may in theory be achieved without the actual prices being equal to the marginal costs. It is enough that all prices be *proportional* to marginal costs; and this could theoretically be the case if every commodity and service were subjected to a tax proportional to its marginal cost. Professor Frisch also suggests the alternative (not seriously as a practical proposal, I think, but as a *curiosum*) that the optimum quantities be achieved by telling the individual that his income tax will be so adjusted that his total tax will be a certain fixed amount. This would be equivalent to deducting from income tax (not merely from taxable income) the other taxes paid. It would seem to have no advantage over simply obtaining all the revenue directly by the income tax and abolishing the excise taxes. And there is a practically insuperable difficulty in that the deductions to be made from income tax would have to be not the nominal taxes levied under the excise law, but the portions (sometimes positive and sometimes negative) falling on each individual. This raises the whole problem of the incidence of taxation on each commodity, which would have to be solved for each individual income-tax return. The incidence of taxation depends in turn on the demand and supply functions of all the commodities in the system, of which a knowledge sufficient for this purpose may be put down as unattainable. Any doubt of the appalling immensity of such a task will be dispelled by a study of the attempts to determine demand functions statistically presented in the substantial and important new book of Henry Schultz.[3]

The practical difficulties of designing taxes proportional to marginal

[1] "The General Welfare in Relation to Problems of Taxation and of Railway and Utility Rates," pp. 242–269.

[2] "The Dupuit Taxation Theorem," this issue, pp. 145–150.

[3] *The Theory and Measurement of Demand*, University of Chicago Press, 1938.

cost are also substantial. Something of an attempt in this direction is represented by general sales taxes. The experience of the American states and cities that have adopted these taxes during the last ten years as emergency depression measures has been far from happy. There is wholesale evasion through buying outside the jurisdiction of the taxing unit, through purchases less than the minimum amount taxable, and through barter. These taxes lend themselves to fraud on the part of retailers, who are designated as agents to collect the revenue from the public; and for a given amount of checking up to prevent fraud, the sales taxes are immensely more costly to collect than income taxes.

These sales taxes, and others that have been proposed, do not even satisfy the theoretical criteria with which we are concerned, for they do not tax services and are not proportional to the marginal cost to the seller. To achieve the end in view, sales taxes would have to be compensated by taxes at the same rate on wages, interest, rents, royalties, professional fees, theater admissions, and other items. The inclusion of wages, rents and interest would give such a tax something of the character of an income tax, but without the graduation by which an income tax takes account of the greater ability to pay of those with large incomes. Furthermore, to conform to the theoretical criteria, the tax must be proportional to *marginal* cost—not average cost, or price, as prices are at present determined; and there must be no other addition to marginal cost. The proposition stands that railways, power plants, and other industries with marginal costs far below the prices charged for their services, could be increased greatly in social efficiency by means of subsidies and reductions of rates. While their rates remain above marginal costs, general reductions of them benefit consumers—in terms of actual money—more than they cost the companies.

The money cost of the inefficiency of a system of excise taxes (of which some may be negative and thus constitute bounties of so much per unit) is definitely measurable in the same sense that the money cost of anything else is measurable. There is only the sort of ambiguity that arises from the effect on the general price level, and thus inheres in any question of the cost of a proposed expenditure; it may generally be disregarded when the cost is not too great in comparison with the total economy. For small excise taxes, if we are content with the assumption of linear functions in a small neighborhood of equilibrium, the approximation obtained on p. 254, namely

$$\tfrac{1}{2}\sum_i \delta p_i \delta q_i,$$

is sufficient. Here δp_i is the increment (positive or negative) in the price paid by a certain individual (or received by him) for a unit of the ith commodity, and $-\delta q_i$ is the increment in the quantity of this com-

modity that he consumes (positive) or produces (negative), as a result of the taxes. The foregoing sum over all commodities then represents the portion of the loss that may be imputed to this individual. The total loss is the sum of such expressions over all individuals. If t_i is the excise tax on the ith commodity and δQ_i the decrement in the quantity of this commodity resulting from the taxes, the total dead loss has the money value

$$\tfrac{1}{2}\sum t_i \delta Q_i,$$

apart from powers and products of the δQ_i, terms that will be unimportant for sufficiently small taxes. This sum is the direct generalization of the little shaded triangle in the classical diagram, Figure 1, p. 243, which has the vertical side t_i and the altitude perpendicular to this side equal to $|\delta Q_i|$. If excise taxes must for any reason be used, they should be designed to minimize it.[4] If by keeping all the taxes strictly proportional to prices, or in any other way, the δq_i can be made zero, it will follow that the δQ_i are zero, and the dead loss is zero.

The fact that the optimum q's are theoretically possible with prices proportional and not equal to marginal costs should not be construed as a defense, even in theory, of the attempt to cover the whole of railway costs, including interest on the original cost of construction, the president's salary, and the rusting of the rails, by levies on the traffic that passes. The system of rates that has evolved out of this attempt is grotesquely disproportionate to the marginal costs even of the railway services. Moreover even proportional additions to the marginal costs for all railway services, without additions in the same proportion to the marginal costs of highway, air, and water traffic, and all other costs, produces social inefficiency of the kind under discussion.

As Professor Frisch remarks, I should certainly not hold that *every* decrease of excise taxes is a good thing. If a toll charge is immutably fixed on a certain bridge, the abolition of a toll on another bridge across the same stream may or may not be a social gain. To decide this question requires a balancing of the losses on account of the traffic that does not cross because of the toll against the uneconomic travel that will be made to and from the free bridge by those who would otherwise find it more convenient to cross by the toll bridge, but go out of their way to avoid paying toll. In general, if a number of competing goods and services are originally taxed, and it is proposed to remove the tax

[4] Criteria for minimum dead loss consistent with a specified revenue have been discussed on pp. 256 and 257 of the July ECONOMETRICA paper, and from another point of view in my earlier paper on "Edgeworth's Taxation Paradox and the Nature of Demand and Supply Functions," *Journal of Political Economy*, Vol. 40, 1932, particularly on p. 607.

from some but not all of them, the question whether the new or the old system is the worse cannot be decided without further data. We can only be sure that both are bad.

We can, however, say that if we begin with an economy in which all sales are at marginal cost, the introduction of a system of excise taxes at varying rates per cent can never produce enough revenue to compensate for the loss of income tax that the state would have to endure if it were to undertake in this way to leave each person as well satisfied as before; provided we ignore the excessively improbable case of all the δq's being exactly zero. This was shown in my paper published in July. Conversely, the abolition of the excise taxes in such an economy will make it possible for each person to contribute more in other taxes to the state while retaining a position as satisfactory to himself as his previous one. And a *proportionate* reduction of all these excise taxes will conduce to the general welfare, though a special discriminatory reduction of one of them may not.

Some commentators on my paper have thought that something must be wrong because the proof that I gave of a system of taxes and bounties diminishing total satisfactions (in the special sense defined) might, it seemed, be reversed to show that after the imposition of the taxes and bounties, an abolition of them would further diminish total satisfactions. A suggestion of this is embodied in the fourteenth paragraph of Professor Frisch's remarks. The fallacy back of the paradox lies in the fact that the proof depends essentially on the original system being one of sales at marginal cost. After the excise taxes and bounties are imposed, sales are no longer at marginal cost; the theorem does not apply to this situation because its premise is not satisfied.

With reference to Professor Frisch's final paragraph, it may be pointed out that the expression $\frac{1}{2}\sum \delta p_i \delta q_i$ for an individual, and the expression $\frac{1}{2}\sum t_i \delta Q_i$ for the whole of the economy, represent correctly the social loss from a system of excise taxes in contrast to more efficient types, *regardless of what the government does with the money*. Whether the government spends the money wisely or badly, and the total amount to be spent by the government, are of course matters of grave public concern. But for a given total of taxation, these measures of the inefficiency of excise taxes, and other additions to marginal costs, are accurate so far as terms of the second order can go.

There is one relevant point which I have failed to bring out, but which is discussed by A. P. Lerner in somewhat similar connections. This is the fact that an income tax of the usual kind is a sort of excise tax on effort and on waiting, as well as on other less defensible ways of getting an income. An income tax is to some extent objectionable because it affects the choice between effort and leisure, and the choice be-

tween immediate and postponed consumption. Thus some of the same kind of loss attaches to an income tax as to excise taxes proper. How serious this effect may be is a question for factual research; but there is some reason to suppose an income tax superior to excise taxes on individual commodities in this respect, even apart from the objectives of progressive taxation and taxation of windfalls and other unearned incomes. To an extent depending on the degree of importance of this substitution of leisure for productive effort because of the income tax, and of immediate for postponed consumption, the public revenues, including those required to operate industries with sales at marginal costs, should avoid income taxes and should be derived primarily from rents of land and other scarce goods, inheritance and windfall taxes, and taxes designed to reduce socially harmful consumption.

Adam Smith, his predecessors, and his immediate successors were concerned with an economy consisting chiefly of agriculture, in which continually rising marginal costs with increasing output are the rule. In agriculture, moreover, there was free competition within national boundaries, with a resultant equality between price and marginal cost. The laissez-faire governmental policy advocated by classical economists was good and efficient under such conditions, apart from the question of the distribution of wealth. It is too frequently forgotten that the conditions of rising marginal cost and free competition are inconsistent with an industrial economy, and that consequently the classical arguments for laissez faire have little application today outside of agriculture and small business. Our theorem includes all that is valid in the essence of the laissez-faire argument, which thus appears as a special case of the more general theory.

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A FURTHER NOTE ON THE DUPUIT TAXATION THEOREM

By Ragnar Frisch

Since there is now complete agreement that Professor Hotelling's mathematical theorem, correctly interpreted, does not single out a tax system with *no* excise taxes, but only the class of tax systems with excise taxes *proportional* to the prices in some base situation, no more comment on this point is necessary.

In my previous note I also tried to explain that there is something very doubtful *about that base situation itself*. Obviously I failed to bring this out clearly enough. On this point I would therefore like to add a few words.

In the usual index-number notation let 0 and 1 be two situations. The change from 0 to 1 we consider as due exclusively to a system of taxes (positive, negative, or zero), $\sum(p_1-p_0)q_1$ is taken as the sum of *the excise taxes*, and $\sum p_0q_0-\sum p_1q_1$ as the change in *income tax*. This definition of the income-tax change is symmetric in the sense that the change from 1 to 0 is the negative of the change from 0 to 1. But this is not true for the excise-tax definition. Here there is a lack of symmetry of fundamentally the same sort as that which exists between Laspeyres' and Paasche's index-number formulae. It is this that gives rise to the tax proposition discussed. If we had defined the excise-tax part of the total tax burden as $\sum(p_1-p_0)q_0$ (the excise-tax sum that would have been paid if the original quantities had been maintained), we would have obtained the opposite proposition ("smaller satisfaction" replaced by "larger satisfaction").

I am not suggesting that one of the above definitions of the "tax burden" is more plausible than the other, but I am saying that whatever definition is adopted, it is important to bring out clearly *the peculiar role played by the base situation*. With this in view the proposition may be formulated as follows: Any tax system whose excise taxes are nonproportional to the prices (p_0) in the base situation *from which we choose to measure "the sum of excise taxes"* [by $\sum(p_1-p_0)q_1$], and whose total levy (income plus excise taxes) is the *same* as that in the base situation chosen [$\sum(p_1-p_0)q_1+\sum p_0q_0-\sum p_1q_1=0$], will yield a *smaller* satisfaction than a tax system whose excise taxes *are* proportional to the base prices and whose total levy is the same as that in the base situation. And all tax systems whose excise taxes are proportional to the base prices, and with the same total levy, will yield the *same* satisfaction. This is only another phrasing of propositions II and IV in my previous note, or, if you wish, of Professor Hotelling's theorem

(when his "no excise taxes" are changed to my "excise taxes proportional to").

A conclusion from the above proposition is of course not invariant for a change in base (as illustrated by the "entropic" example in my previous note). The meaningful application of the proposition therefore depends on the possibility of determining an "archaic" base price set that in some sense of the word can be considered as *not affected* by taxes or by any government action indirectly related to taxes.

Professor Hotelling takes "prices at marginal cost" as such a set. In my view this is not justified. When it comes to *applications* to a modern society where all sorts of monopolies and quasi-monopolies exist, (i.e., producers not quantity adapters) the equality between prices and marginal cost cannot be taken as a criterion for the absence of excise taxes, and still less for an "archaic" situation unaffected by other taxes or other government action. (For the consumers, quantity adaptation is a more plausible assumption.)

However this may be, it should be clearly recognized that the mathematical analysis itself as presented by Professor Hotelling (or by me) *does not contain any argument for the selection of marginal cost as the base price set.* As a matter of fact the concept of marginal cost does not enter into the proof at all, as everybody can check by going through the mathematics. The mathematical theorem applies equally well to any other price system which for one reason or another we may adopt as the base for our comparisons.

I would therefore be inclined to formulate the practical conclusions —and in particular those presented as proved mathematically—in much less categoric terms than those used by Professor Hotelling.

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Oslo

A FINAL NOTE

By Harold Hotelling

The proof of the fundamental theorems of my paper was on the basis of a set of assumptions including one to the effect that prices in the absence of the excise taxes were equal to marginal costs. It follows quite readily from these theorems that the attainment of the maximum of the general welfare, in the only sense that can apparently be given to this expression with the help only of rank ordering of a person's satisfactions without interpersonal comparisons, requires that all sales be at such prices that the quantities bought will be exactly the same as if these sales were at marginal cost. In particular, in considering what prices should be charged by state-owned or regulated enterprises, we have marginal cost as the criterion, leaving no place for the criteria generally used, such as average cost.

In the proof the change δr in government revenue from a person resulting from the imposition of a set of excise taxes, with a decrement δm in his income tax, was expressed (p. 250) by

$$\delta r = \sum q_i' \delta p_i - \delta m,$$

where δp_i is the increment in the price of the ith commodity resulting from the change, q_i' is the quantity this person buys of the ith commodity after the change, and $\sum$ denotes summation over all commodities. I showed that the change from income to excise taxes makes δr negative if the person's satisfaction is left unchanged; and that if, on the other hand, the rates are such that the revenue is unchanged, so that $\delta r = 0$, the person's satisfaction is diminished.

Now Professor Frisch's argument claims to apply to this a *reductio ad absurdum* by showing that any "self-financed" change, i.e., any change for which $\delta r = 0$, diminishes the person's satisfaction, even if prices were not originally equal to marginal costs. It is true, as both he and I have now proved, that every change making $\delta r = 0$, and not leaving all the quantities unchanged, reduces the person's satisfactions. His "entropic" interpretation seems to imply that changes in general make things go from bad to worse, that we had better endure present evils than fly to any other possible situation whatever, and that it is not only from a state of sales at marginal costs that changes are objectionable. He feels that in my proof no actual use has been made of the assumption of marginal cost being initially equal to price and that some undefined archaic system may be preferable to this social goal.

The key to this paradox lies in the interpretation of the condition $\delta r = 0$. This condition provides a suitable definition of a self-financed

change *only if prices are originally equal to marginal costs*, or if the quantities sold are exactly as if prices were equal to marginal costs. If sales are originally at prices different from marginal costs, the differences constitute a revenue (positive or negative) whose magnitude will in general change as a result of the change in the price and tax system. Such a change in revenue must be considered in addition to the expression above for δr. It is not a matter of indifference to a government tobacco monopoly, for example, if the demand for tobacco is reduced by reason of a special tax on railway tickets required for riding in smoking compartments—unless the tobacco monopoly is selling at marginal cost.

Reverting to the case of a publicly-owned toll bridge with which I began my original paper, let us consider that all costs resulting solely from use of the bridge are zero. The marginal cost of crossing is thus zero. Assume that a toll charge p (not regarded as a tax) is made per crossing, and that, at this rate, q crossings are made per year. There is thus a public revenue pq from this source. Whether this revenue is or is not required to pay interest on the cost of the bridge is not relevant to the present question. Now suppose that a tax t per crossing is levied on each user in addition to the toll; and that, because of the tax, the number of crossings per year thereafter is reduced to q'. In the foregoing expression for δr there is only one term $q't$ in the summation, in which δp_i is replaced by t and q_i' by q'. If we were to regard a measure as self-financed when $\delta r = 0$ and if the entire proceeds $q't$ of this tax were used to abate the income taxes of the persons crossing, we should have to call this a self-financed measure. The weakness of such a definition of self-financed measures is revealed by the fact that the government will not be in so good a position as before; for the reduction of the number of uses from q to q' will reduce its ordinary revenue pq to pq'. If we supposed that the combination of the tax on bridge crossings with reduction in income taxes really left the treasury in as good a position as before we should have to have $pq' = pq$. This would imply either that $q' = q$, i.e., that the increased cost of crossing had no effect on the traffic, or that $p = 0$, i.e. that the price charged for the use of the bridge was only the marginal cost. Since the possibility $q' = q$ is excessively improbable, we are forced to the conclusion that only when the original charge was the marginal cost is it justifiable to regard as self-financed those changes satisfying the condition $\delta r = 0$.

Likewise in general the condition $\delta r = 0$ has real significance only where the sales are in the first instance those corresponding to prices equal to the respective marginal costs. It is in this way that the argument depends essentially on the concept of marginal cost. It is necessary in demonstrating the loss of satisfactions to confine ourselves to

changes that leave the total budget in balance—including revenues from tolls, postage, licenses, government-owned railways, etc.—and not merely the budget derived from revenues ordinarily called taxes.

It is in this way that the foregoing theorems, and the practical conclusions drawn from them in my paper regarding the efficiency of operations at marginal cost, have a significance that actually involves marginal cost in an essential way.

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REPORT OF THE CRACOW MEETING, SEPTEMBER 18, 1938

By B. Debiński and J. Wiśniewski

The eighth European meeting of the Econometric Society was scheduled for September 18–21, 1938 at Cracow (Kraków), Poland. Unfortunately, the political tension existing at that time prevented many members from attending. Polish members, on the other hand, were fairly numerous. The number of those present, including guests, totalled 39.

The meeting lasted only one day, September 18. In the morning Dr. Wiśniewski read his paper, the summary of which is given below; and in the afternoon Professor Bilimovic's paper was read from a summary which he had sent to Cracow. Professor Bowley and Professor Taylor of the University of Poznań presided at the sessions, which were held in the premises of the Chamber of Commerce. Professor Zawadzki and Dr. Ziomek were in charge of the organization of the meeting. A dinner was given by the Cracow Economic Society.

A summary of the papers and of the discussion follows.

Jan Wiśniewski, Academy of Commerce, Warsaw: *The Distinction between Metric and Symptomatic Indices and Its Bearing upon Production Indices.*—Professor Wiśniewski defined a metric index as a sequence of numbers that fulfill two conditions: first, they presumably represent an economic phenomenon that is thought of as a measurable quantity; second, there is a definite and known functional relationship between the index and the magnitude of the economic phenomenon. More often than not, this relationship is a simple proportionality, whose constant may remain unknown. The functional relationship may be fulfilled rigorously or within the margin of sampling error. A symptomatic index is a sequence of numbers that presumably represent an economic phenomenon but fail to fulfill at least one of the above conditions. It is not difficult to find examples of both kinds of indices. We often meet with indices of economic phenomena that are not, or, indeed, cannot be defined as measurable quantities. Such indices are, of course, symptomatic ones. Let us mention only "general business conditions." Unless we decide, more or less arbitrarily, to give these words a more precise definition, metric indices of such a phenomenon are impossible to construct. A case where such definitions are being given is the general level of prices or cost of living. As a matter of fact, as soon as we decide to compute an index according to a given formula and to call it "the" index of prices or of cost of living, we adopt a specific definition of the general level of prices or of cost of living. The case of indices of production is very nearly the same. "Production" is an unequivocal concept provided we deal with the production of a single and simple commodity, expressed in physical

units. The situation grows more complex when we have to do with a more or less comprehensive group of commodities. A quantitative definition of "volume of production" becomes unavoidable. Economic and statistical theory was hitherto much less interested in production than in price indices. Divisia's and Fisher's theories may be mentioned, having in fact very much in common. Their underlying assumption is that prices and quantities (quantities exchanged or quantities produced) should be treated symmetrically. The (new) Polish index of industrial production which Professor Wiśniewski was charged to construct is a deliberate attempt to approximate a metric index built along the lines of the Fisher-Divisia theory. The construction of this index is more amply described in a paper (with an English summary) published in No. 3–4, 1938, of the Polish journal *Przegląd Statystyczny* (*Statistical Review*).

The discussion which followed Dr. Wiśniewski's paper was concerned with the principles and details of his index.

ALEKSANDER V. BILIMOVIC, Ljubljana, Jugoslavia: *Einige Bermerkungen zur Theorie der Planwirtschaft* (*Some observations on the theory of planned economy*).—Professor Bilimovic's paper discussed the problem whether the thorough collectivist economy represents a determinate system. The comparison of the number of unknowns with the number of equations of the system proves that the planned economy does not represent the definite system because of three elements, which may be quite arbitrarily decided by the state: (1) The problem of each individual's share in the social dividend. The social dividend consists of the remuneration of land and interest on money capital; since these are the common property of the whole society, there exists a necessity for the planned economy to introduce some rule of distribution of income among individuals. (2) The second loose point is caused by the nonexistence of a real market for capital goods and therefore the arbitrary choice of the rate of interest by the decision of the state. (3) But the most fundamental difference between the competitive economy and the planned one consists in the fact that the real forces, which in competitive economy influence prices and costs, tending to equalize the first to the second, are completely lacking in the planned system. The state may quite deliberately decide to bear losses on production of some commodities and realize profits on others and nothing can forbid it to do so. The system will become solvable if we assume that the state aims at maximum of utility for all individuals but such an assumption implies the summation of individuals' satisfaction functions. Though this is impossible from the scientific point of view, in practice it is connected with the problems of social policy and public finances. The nonexistence of any rational criterium for summing the

individuals' satisfactions is the reason for the almost unlimited freedom of authorities in deciding the problem. The danger of a completely collective system lies in the fact that the state possesses all means of introducing its decisions by force and there does not exist any counterbalancing power. The role of practical policy will be the optimal choice between unlimited freedom of the individual and unrestricted arbitrariness of the state. The difficulty consists in the fact that such intermediary hybrid forms are always tending to extremes in one direction or the other.

Professor ARTHUR L. BOWLEY asked whether the system described by Professor Bilimovic took into consideration the question of individuals' time-preference, which would have influenced the rate of interest. Mr. BOLESLAW DEBIŃSKI of the Academy of Commerce, Warsaw, who read the summary of the paper in the absence of the author, explained that Professor Bilimovic based the whole reasoning on the existence of long-period equilibrium and did not analyse nor even mention the problem raised by Professor Bowley.

Professor ADAM HEYDEL, Jagiellonian University, Cracow, discussed the problem of the value of capital goods on the lines of Dr. Oskar Lange's book. He doubted possibility of defining capital-goods prices in an economy in which these goods would be *extra commercium*, out of exchange. The example of the possibility of computing the value of capital in industries merged vertically, which sell on the market only consumption goods, might suggest that this possibility would exist even in a collectivist system. Some people may point out that there exists always in the capitalistic system the alternative of selling capital goods or using them in production. This alternative makes it possible to calculate in an indirect way the value of capital goods. This possibility is excluded in the system proposed by Dr. Oskar Lange. On the other hand the relation of the real prices of capital goods and consumption goods might be deducted from the amount of labour necessary for production of capital goods. Wages received by workers mean a certain amount of consumption goods; at the same time they define the cost of a unit of capital goods. There arises nevertheless another difficulty according to Dr. Lange: (1) all wages should be equalized; (2) the rate of capitalization should depend on decisions of authorities directing the collectivist economy. The wage unit would then be an arbitrary magnitude and would correspond to an arbitrary part of the social dividend, divided by the number of workers (working days). It seems doubtful whether such a unit might serve as a base for the valuation of capital goods.

Professor EDWARD TAYLOR, University of Poznań, pointed out in reply to Professor Heydel that the analogy between a vertically con-

centrated industry and the thorough collective system is doubtful, as not only the existence of market prices, but the problem of multiplicity of independent producers with different aims is important—that manifoldness of aims is excluded in the planned economy where the state has to decide everything and later only some corrections may be introduced. Oskar Lange's assumption of the possibility of preservation in the planned economy of the multiplicity of aims of different producers by the decentralization of the control of industry is incompatible with the fundamental principles of collective system.

Professor JANUSZ LIBICKI, University of Poznań, expressed some doubts whether the preservation of the market for labour and exclusion of all other markets in the planned economy will not cause some unavoidable contradictions and serious difficulties in adjustment of demand for and supply of different kinds of labour. Under the system of free competition the changes tending to the point of equilibrium must occur; in the planned economy they may occur or may not.

Professor ADAM HEYDEL stressed that Dr. Lange's plan does exclude the existence of monetary savings made by individuals. He agreed with Professor Zawadzki that the main advantage of the competitive economy lies in the fact that the entrepreneur's errors mean his losses. This fact leads to an automatic reaction which corrects errors. This automatic reaction can scarcely be replaced by governmental measures and sanctions in the planned economy.

Professor WLADYSLAW ZAWADZKI,[1] Academy of Commerce, Warsaw, pointed out that it is impossible to defend the planned economy, arguing by comparison that perfect competition does not exist under present-day conditions either. We cannot apply the results of analysis of real systems to state or to suggest what would happen in the abstract ideal conditions, but we have to analyse those abstract conditions themselves. The central problem of the paper consists in the question whether in collectivist economy there does exist any movement towards any definite point of equilibrium, any tendency of change in the direction of equilibrium. Pareto amplified the Walrasian system of equations of equilibrium having included in it the existence of monopoly, but did not solve the problem in case of existence of a larger number of monopolies. There must occur serious difficulties in the determination of coefficients of production in consequence of the lack of forces existing in the competitive economy. The monopolist does not possess any means of ascertaining these forces. The same difficulty has to occur even in a higher degree in the collectivist economy, where the forces aiming at mutual adjustment of costs and

[1] As this issue goes to press, we regret to learn of the death of Professor Zawadzki on March 8, 1939.—MANAGING EDITOR.

prices are lacking. Therefore we are unable either to imagine or to realize how the planned economy will obtain the optimal coefficients of production. The path of successive approximations is completely unknown. We have no index showing that the system does not reach the optimal position; the real forces which secure the determinateness of competitive economy do not exist in the planned economy. The economic plan of the state must be based on the maximisation of total utility for all individuals, related to a certain distribution of income. Both Bilimovic and Lange reject the introduction of the compulsion of labour and aim at the preservation of a free market for labour. There are serious doubts if this possibility of a free market for labour is logically tenable, but, if we admit it is, it complicates the solution of the system and must veil errors in the determination of coefficients of production. The examples of Russia and Germany with the increasing numbers of workers bound to factories on conditions of forced or even slave labour do not prove the possibility of a free market for labour in the planned economy.

Mr. ANTONI TOM, Ministry of Agriculture, Department of Economic Policy, said that an economic system with completely socialized means of production and the coexistence of consumers' free choice can only by chance attain the equilibrium position *immediately*. Therefore it is impossible to speak about *planned* economy in which the execution complies with the previously established plan. It is possible to speak about *managed* economy only (the equilibrium will be attained by successive trial and error) which is not identical with the system described by the socialist economists. The Board of Planning having fixed the prices of different consumers' goods, for instance on the base of prices inherited from the capitalist system, sees *ex post* whether the equilibrium in the market exists, i.e., commodity stocks do not increase nor do prices rise. Accumulation of stocks of a certain good proves that either its price is too high or its supply too large. On such a basis it is possible to introduce adequate changes, when one knows the results of statistical enquiries into elasticities of demand for different goods. The freedom of consumers' choice is to a certain extent limited by the fact that there do not exist goods more desirable in the *absolute* meaning, therefore the state can consciously shape individuals' preferences by fixing primarily certain relations of prices. In a similar way one may see whether total quantity of consumers' goods produced is adequate or not. Too small demand causes the monetary savings, which nevertheless does not threaten the collapse of the system as long as the prohibition of privately managed production is respected. On the other hand, in this case the principle of equal opportunity for all individuals cannot be longer maintained. When all production goods

belong to the state and various combinations of production factors are indifferent, the equilibrium on their market always exists. This last statement is subject to certain limitations and restrictions.

Warsaw, Poland

REPORT OF THE DETROIT MEETING, DECEMBER 27–30, 1938

THE AMERICAN winter meeting of the Econometric Society was held at Detroit, Michigan, December 27–30, 1938, in connection with the meetings of the other social-science societies. All the sessions were held in the Book-Cadillac Hotel, which was the headquarters of the Society.

Announcement was made during the sessions regarding the meeting of the International Congress of Mathematicians, which is to be held in Cambridge, Massachusetts, September 4–12, 1940. Persons interested in further information should address R. G. D. Richardson, Secretary of the Congress, at Brown University, Providence, R. I.

The opening session, on Tuesday morning, December 27, was devoted to the Dynamics of Automobile Demand.* The chairman was Colonel Leonard P. Ayres of the Cleveland Trust Company, and the following papers were presented:

SEWARD L. HORNER, General Motors Corporation, *Statement of the Problem.*—In a paper which represented a statement of the problem (to be dealt with in greater detail by Messrs Roos, von Szeliski, and DuBrul in succeeding papers) Mr. Horner discussed the factors which influence fluctuations in the demand for automobiles. By the demand for automobiles in these studies is meant the sales of new passenger cars to consumers in the United States. This is a derived demand, since the primary demand is for individual transportation service. But transportation service may be obtained for longer or shorter periods without purchasing a new car, since the automobile is a durable good. The two types of demand should be clearly distinguished. Some of the factors which have influenced the demand for transportation service, and thus the long-time trend of new-car sales, are the improvement in the product and the lowering of real price, the decrease in the cost of operation, the increased mileage of good roads, the greater availability of service, the increase in population, and improved standards of living. In these studies, however, major attention is directed to factors operating over a shorter period. The more important factors influencing the year-to-year fluctuations in new-car sales are believed to be general business activity and national income (considering both the level and direction of change), the cost of living in relation to national income, new-car prices, used-car prices and trade-in allowances, the number of cars in operation, and the age of cars. Factors which influence more particularly the month-to-month changes in sales are seasonal variation, new-

* The papers on this subject presented at the Tuesday morning and afternoon sessions are being published in pamphlet form by the General Motors Corporation.

model stimulus and date of announcement, the trend of general business activity and national income during the year, field stocks of new cars, and used-car stocks. While it is necessary to deal with all of these factors, and while a fairly good statistical explanation of the fluctuations in sales may be obtained without using price as a variable, the primary purpose of these studies is to attempt to isolate and measure the effect of price and price changes upon the demand for automobiles.

CHARLES F. ROOS, Institute of Applied Econometrics, *Factors Governing Yearly Changes in Domestic Automobile Demand.*

VICTOR VON SZELISKI, Institute of Applied Econometrics, *Additional Factors Governing Monthly Changes in Domestic Automobile Demand.*—In these two papers which were really the presentation of a joint study, Dr. Roos and Mr. von Szeliski said that satisfactory statistical demand functions for durable goods have generally been difficult to derive. A new approach to the problem is required which will take direct account of the most outstanding characteristic of durable goods, viz., durability. The consumers' inventory, which is not important in the case of perishable goods, is in large part responsible for the major cyclical moves in automobile sales. Other important determinants are the supernumerary income (income payments less direct taxes less necessary living costs), price, and used-car allowances. A one-per-cent increase in supernumerary income will produce about a $2\frac{1}{2}$-per-cent increase in new-car sales. A one-per-cent increase in price will cause a decrease of from 0.75 to 2 per cent in sales. A one-per-cent increase in the ratio of trade-in allowance to new-car price will cause a little less than 0.5-per-cent increase in new-car sales. A one-per-cent decrease in the average annual depreciation rate (price of the automobile divided by years of its life) will increase the maximum-ownership level by about 0.3 per cent. Maximum-ownership level is defined as the number of automobiles that the population would support at any one time and under any one set of economic conditions. A one-per-cent increase in supernumerary income would raise the maximum-ownership level by about 0.25 per cent. A one-per-cent increase in supernumerary income would increase replacement sales by about 0.7 per cent. As a result of the introduction of all-steel construction and other factors, the life of the average motor car has been increasing and is now nine or ten years. The two principal factors determining the monthly distribution of sales are the new-model stimulus and the "pure" seasonal. By using dates when new models of individual makes were introduced, it is possible to analyze the observed deviations of retail sales from a smooth cycle line in terms of a constant seasonal function and a new-model stimulus varying with the date of model introduced and the percentage of the industry introducing models on specific dates. A combination

of the short-term with the long-term factors of cars in operation, national income, and price, results in an almost complete explanation of the observed retail sales. The finding of a price elasticity for new-car sales of less than 2 and possibly as low as 0.75 is at first sight at variance with the undoubted non-necessary status of automobiles which would seem to indicate a very high elasticity. The answer lies in the durable character of automobiles, and in the fact that consumers have, over the short term, other ways of obtaining their automobile transportation. They can run their existing cars longer with somewhat higher operating maintenance costs, or they can obtain cars from dealers' used-car stocks. Consumers thus have access to a considerable source of supply independent of new car production and, over the short term, at a substantially lower outlay. Hence when other factors are unfavorable, price cuts can fail to stimulate sales.

The subject of the Dynamics of Automobile Demand was continued at the Tuesday afternoon session, with Professor Theodore H. Brown of Harvard University in the chair. The papers were as follows:

ANDREW T. COURT, Automobile Manufacturers Association, *The Problem of Measuring Automobile Prices and Values.*—Mr. Court discussed "hedonic" commodity price indexes. He contended that no price index can be constructed without price comparisons between periods of time. Valid comparisons necessitate adequate commodity standards, i.e., definitions of the articles priced. In the case of complex, rapidly evolving goods, serious errors result from nominalistic identifications based on brand name or other inadequate standards. Among semi-adequate techniques for establishing continuous price comparisons which have serious shortcomings are: 1. Averaging prices of products falling within specification ranges; 2. Overlapping quotations; 3. Price per unit specification. The hedonic suggestions make possible price comparisons on a realistic, objective basis in the face of rapidly changing product design. The *hedon* is the (still undefined) unit of usefulness or desirability. Although the *hedon* is a unit reflecting subjective qualities, it can be established objectively on an approximate and relative basis as a composite of the specifications (reflecting useful qualities) describing any class of products. The weights used in combining specifications are determined by means of a multiple-regression analysis of the relation between prices of various articles offered at any one time and the specifications reflecting the useful qualities of those articles. A second hedonic approach computes directly the net regression of prices on time, holding usefulness and desirability (as reflected by relevant specifications) constant. The validity of any hedonic analysis is dependent upon securing specifications for individual products which actually reflect their useful and desirable properties. Average measure-

ments in the trend of the following car qualities from 1926 to 1938 are presented for the first time: 1. Comfort; 2. Performance; 3. Durability and dependability; 4. Economy.

STEPHEN M. DUBRUL, General Motors Corporation, *Significance of the Findings.*—Mr. DuBrul said that an analysis of the financial statements of the various automobile manufacturers shows conclusively that the margin between the "avoidable" cost of production per unit and selling price per unit is so narrow that even if the elasticity of demand for automobiles were as great as 2, it would be impossible to increase volume by cutting prices under these conditions without inviting insolvency. A policy of price adjustment that would be designed to lower prices when the intensity of demand has shrunk would have to be accompanied by a policy of increasing prices sharply in periods of active demand in order to accumulate the required reserves with which to cover the losses in the periods of reduced prices. Since this is impossible competitively, it would necessitate the reorganization of the industry on a cartel basis. The facts of the preceding analyses indicate also that the elasticity of demand with respect to price in the case of durable goods actually may be zero in periods of falling national income and consequently a price adjustment would be ineffective in stimulating demand. This is simply another aspect of the law of diminishing marginal utility. The analyses also brought out the tremendous necessity in the automobile industry of maintaining its previous rate of progress in order to maintain its demand, because if progress is not maintained the life of the product will be prolonged and the current demand will fall. The fact also stands out clearly that analyses of this type are tremendously involved and much work remains to be done in the field of methodology before simple answers can be obtained.

At the Tuesday evening session Problems in Industrial Replacement and Standards of Quality were considered. Dr. Robert W. Burgess of the Western Electric Company presided and the following papers were given:

WALTER A. SHEWHART, Bell Telephone Laboratories, *Importance of Some Statistical Characteristics of a Standard of Quality.*—Dr. Shewhart pointed out that many business economists and statisticians are called upon to assist the engineer and industrial scientist to secure the full economic and social advantages of using standards of quality in the production and exchange of manufactured goods. In order to attain the most effective industrial and social use of such standards, the engineer must provide an accurate description of what constitutes a standard of quality and of how such a standard may be established and maintained in the face of ever-present unknown or chance causes

of variability. The present paper attempts to provide such a description. Whereas it is usually assumed that a standard of quality may be fixed by specification, such an assumption may easily be proved unjustifiable in terms of operationally verifiable meaning. Instead, a specification of quality is inherently a statistical hypothesis and as such can never be verified with certainty. In purchasing, therefore, a consumer must choose not between goods whose quality is known with certainty but rather between goods whose quality is known with different degrees of probability based upon evidence obtainable in one or both of the steps of production and inspection. The preparation of a specification constitutes the first of the three co-ordinate steps—specification, production, and inspection—in mass production. Inspection, involving the act of judging whether or not a thing is of specified quality, helps to shape the standard in much the same way that the action of a judge helps to shape and give operational meaning to a statute. An efficient mass-production scheme requires that these three co-ordinate steps be designed so as to make the most efficient use of the results obtained in the second and third steps. Usually the most economical and satisfactory standard of quality is one produced under a state of statistical control such as may be reached as a statistical limit in a continuing process of mass production. In conclusion we may say that an economic standard of quality is not a written finality but a dynamic process in which modern statistical theory plays an imporant rôle. It is not merely the imprisonment of the past in the form of a specification but also the unfolding of the future as revealed in the process of production and inspection. These facts must be taken into account in the production and exchange of economic goods if the most economical use of raw materials in the satisfaction of human wants is to be attained.

J. S. Bain, Harvard University, *A Comment on the Theory of Reinvestment Cycles.*—Mr. Bain remarked that there are roughly two views regarding the cycle in reinvestment in fixed equipment. One school holds that this is a "pure cycle" which once begun can exist independently of the cycle in general business and can act as a generating force accounting for turning points in the business cycle; the other school holds that the cycle in reinvestment is "secondary," or generated by the cycle in general business, and depends upon the latter for its continuation. The theory of the pure cycle rests upon the assumption that the economic life of equipment may for practical purposes be taken as a parameter in the analytical scheme; the theory of the secondary cycle, if fully interpreted, recognizes that the economic life of equipment is a variable. It is contended here that the economic life of industrial equipment is a variable sufficiently sensitive to changes in

the economic environment which occur over the business cycle so that any cycle in replacement is likely to be of the "generated" rather than the "generating" sort; further, that from this it follows that turning points in the business cycle are with no certainty related to or caused by the termination of the lives of items of industrial equipment.

ALFRED J. LOTKA, Metropolitan Life Insurance Company, *Contribution to the Analysis of Industrial Replacement.*—Dr. Lotka stated that the analysis of problems of industrial replacement form part of the more general analysis of problems presented by "self-renewing" aggregates. Certain equations to which such problems lead are identical in form when relating to population and industrial equipment. The particular case of a body of policyholders maintained constant in number by the continual accession of new members to replace the charter members dropping out by death was first considered by L. Herbelot in 1909, who obtained the integral equation since then discussed by many writers both in population studies and with reference to industrial replacement. Herbelot and several authors following him applied the method of successive differentiation in the solution of his fundamental equation. A different treatment, leading to a series of exponentials, was developed by P. Hertz and G. Herglotz in a problem of electrodynamics, and was subsequently applied by F. R. Sharpe and Alfred J. Lotka to population problems, and by Alfred J. Lotka to the problem of industrial replacement. In the present communication, after an historical introduction, some details were further elucidated, and the advantages and disadvantages of the several methods employed by different authors were set forth.

On Wednesday morning, December 28, The Behavior of Prices was discussed. Professor Harold Hotelling of Columbia University was in the chair, and the following papers were given:

OSKAR LANGE, University of Chicago, *Say's Law: A Restatement.*—Say's law is the proposition that there can be no excess of total supply (general overproduction) because total supply is identically equal to total demand. Professor Lange proved that if n commodities are exchanged, one of them being used as money, the sum of the n supply functions is identically equal to the sum of the n demand functions. Let the demand functions be $D_i(p_1, p_2, \cdots, p_{n-1})$ and the supply functions $S_i(p_1, p_2, \cdots, p_{n-1})$, where $p_1, p_2, \cdots, p_{n-1}$ are the prices of the $n-1$ commodities and $i=1, 2, \cdots, n-1$. The supply function of the nth commodity, which functions as money, is $S_n=\sum_{i=1}^{n-1} p_i D_i$, and the demand function for money is $D_n=\sum_{i=1}^{n-1} p_i S_i$. Hence we obtain $\sum_{i=1}^{n} p_i S_i \equiv \sum_{i=1}^{n} p_i D_i$, i.e., total supply identical with total demand. This may be called Walras' law. But if we exclude money from among our commodities the identity does not hold any more. $\sum_{i=1}^{n-1} p_i S_i =$

$\sum_{i=1}^{n-1} p_i D_i$ only when $D_n = S_n$ (monetary equilibrium). Say's law, however maintains that supply of all commodities, money excluded, is identically equal to their demand, i.e., $\sum_{i=1}^{n-1} p_i S_i \equiv \sum_{i=1}^{n-1} p_i D_i$. The necessary and sufficient condition that this be true is $D_n \equiv S_n$. This is equivalent with postulating that money functions only as a medium of exchange and not as a store of value, i.e., that people do not want to change their cash balances. This was the assumption of the classical economists (money only a "veil"). As shown in further detail in the paper, this assumption, however, does not imply that there need be full employment or that total demand covers total cost at any level of output. It is further shown that the acceptance of the identity $S_n \equiv D_n$, which underlies Say's law, makes determination of the prices impossible. In the general case the prices $p_1, p_2, \cdots, p_{n-1}$ are determined from the $n-1$ independent equations: $D_i(p_1, p_2, \cdots, p_{n-1}) = S_i(p_1, p_2, \cdots, p_{n-1})$, where $i = 1, 2, \cdots, n-1$. But if $S_n \equiv D_n$ there are only $n-2$ independent equations and only the ratios of the prices (relative prices) can be determined. In such case it also can be shown that these ratios and the quantities of goods exchanged (and produced) are independent of the supply of money S_n (i.e., money is neutral) and that it affects only the general price level. If S_n could be treated as a datum the latter could be determined. S_n, however, is not a datum, for it is not the quantity of money but a stream of money expenditure MV (M = quantity of money, V = velocity), and the decomposition of $S_n = MV$ into M and V is indeterminate. This indeterminateness disappears when the identity $S_n \equiv D_n$ is replaced by an equation $S_n = D_n$ (equation of monetary equilibrium). This implies giving up Say's law and the neutrality of money. The supply of money affects not only the general price level but also relative prices and the quantities of goods produced and exchanged. General equilibrium theory and monetary theory cannot be separated.

CARVEL LANGE and CHARLES F. ROOS, Institute of Applied Econometrics, *The Dynamics of Commodity Prices.*—Mr. Lange and Dr. Roos pointed out that commodity prices are the result of all economic factors. Nevertheless, the most important factors are those associated with demand and supply of the commodities and money. Demand seems to be the dominant factor. In fact, one can achieve a high percentage of accuracy in anticipating price changes by measuring unbalanced situations with respect to demand alone, neglecting supply entirely. Suitable measures of demand are: (1) the relation of new orders obtained by manufacturers to consumers' spending; (2) the total spending for goods and services; that is, debits less speculative and financial transactions, wages, and imports; (3) the rate of change of bank deposits. Foreign demand is reflected by foreign exchange rates even under

so-called managed money economies. Speculation merely intensifies a trend generated by the basic economic factors but without changing its direction. The same in general is true of politics. But political action with respect to monetary policy does have important effects and can be helpful. In their work Mr. Lange and Dr. Roos assumed that government policies could be entirely ignored, except as above noted, because their studies showed that indexes of orders and production pointed the direction of the price move and that the Government, if it did anything at all, merely contributed to the move already indicated. Dr. Roos presented a correlation which he and Mr. von Szeliski worked out showing that the factors so far discussed can explain about 95 per cent of the monthly variation in commodity prices. Since 1919 the correlation series actually leads prices by two to three months.

EMIL LEDERER, New School for Social Research, *Can the Structural Trend be Measured?*—Professor Lederer stated that the problem of the structural trend is related to though not identical with the secular trend. The problem is to find this part of Investment and to measure it quantitatively which is—because of its dynamic force—the center of the expanding tendencies within an Economic system. An Index of Investments in New Industries could serve this purpose. New Industries imply investments in other Industries ("inner multiplier"), which however is different as to: Perishable Consumers' Goods, Durable Consumers' Goods, Durable Producers' Goods, Means of Transportation. An analysis of prospects for New Industries would determine the strength of these expanding forces, in the absence of which investments after a period of recovery would hardly take place, either at the prevailing or at a lower interest rate. Studies of the past as well as an investigation into the nature of impending investments could clarify the question whether or not such an Index could be constructed. It would be an attempt at qualitative Investment Analysis, which would not take Investment indiscriminately as an aggregate, but would investigate it in its relation to expansion.

The Wednesday afternoon session was devoted to Problems of Industrial Structure, under the chairmanship of Professor Arthur W. Marget of the University of Minnesota. The papers were as follows:

JOHN M. CLARK, Columbia University, *One Form of Price Competition between Geographically Separated Large Producers.*—Professor Clark showed that with geographically separated producers of a standardized commodity, selling under the rule of a uniform price at the mill plus actual freight, each will be the only seller in a given area, the size of which is determined by his price and freight costs, relative to those of surrounding producers (with a possible border zone reached by more than one producer). Except for this zone, the individual de-

mand schedule for a given producer is, in principle, computable, and has a finite slope, making it possible to compute under simplified conditions the level to which competition will bring prices if producers are not governed by the expectation that their price changes will be met by rivals. Under representative conditions of marginal cost, total cost, freight rates, and excess capacity, such competition tends to prices above marginal cost, but considerably below total cost. Variant conditions include irregular distribution of demand, differences in the slope of the individual demand schedule for different producers, and changes in this slope at critical points with changes in the relative prices of different producers. Practical considerations affecting the outcome include: (1) The virtual certainty that price reductions will be met, tending to lead to an indeterminate stalemate in prices, with possibly greater stability than at present. (2) Irregular and secret price cutting (usually discriminatory) would be eliminated or greatly reduced, thus removing or greatly weakening one competitive force which is of great importance under existing conditions.

MAX MILLIKAN, Yale University, *Comments on the Duopoly-Oligopoly Problem.*—Dr. Millikan said that in analyzing the oligopoly problem we must distinguish between those cases in which the policy of sellers operates on price, which, following Mr. Kahn, we may call the Bertrandesque case, and those in which it operates on quantities, which we may term the Cournotesque case. Limiting ourselves to the Bertrandesque case and abstracting from problems of time, product variation, and selling costs, joint products, and discrimination, we may say that (in the absence of collusion or agreement) oligopoly exists if any seller regards his sales as a function of at least one other price besides his own. That is, if his conjectural demand function is of the form: $x_1 = x_1(p_1, p_2, \cdots, p_n)$ where x is sales and the p's are prices and where n is equal to or greater than 2, we may then find an expression for the equilibrium position of any producer by differentiating his profit (expressed as a function of his price, his sales, and his costs) with respect to his price and setting the resulting expression equal to zero. In the duopoly case, this gives the expression:

$$\frac{\bar{d}\pi_1}{\bar{d}p_1} = x_1 + (p_1 - \gamma_1)\left(\frac{\bar{\partial}x_1}{\bar{\partial}p_1} + \frac{\bar{\partial}x_1}{\bar{\partial}p_2}\frac{\bar{d}p_2}{\bar{d}p_1}\right) = 0$$

where γ_1 is marginal cost and the bar is placed over the derivative operators, total and partial, in order to indicate the conjectural character of these quantities. Any duopoly solution may then be deduced by substituting in appropriate values of the derivatives. It is to be noted that attention has been largely concentrated up to now on the

conjectural variation of one price with respect to another, i.e. $\bar{d}p_2/\bar{d}p_1$. It should be recognized that the partial derivatives, $\bar{\partial}x_1/\bar{\partial}p_1$ and $\bar{\partial}x_1/\bar{\partial}p_2$, which depend on the market interrelations of demand between the two commodities, are at least equally important and probably more so since the conjectural derivatives themselves probably depend in large measure on the market structure reflected in these partials. It is normally assumed, for instance, that a change in the price charged by one producer will always induce a change in the same direction in the price charged by another producer who takes account of the first's action. If the conjectural variation of one producer is zero a rise in a competitor's price will only induce a rise in his own price under certain types of market situations. If we define Bertrand demand for any producer as the partial-demand curve which results if all other producers' prices are held constant, then the first partial, $\bar{\partial}x_1/\bar{\partial}p_1$, represents the slope of the Bertrand demand curve at the optimum point and the second, $\bar{\partial}x_1/\bar{\partial}p_2$, the rate at which the Bertrand curve shifts to the right or left with changes in a competitor's price. If a firm's conjectural variations are zero, its equilibrium position is determined entirely by the position and slope of its Bertrand demand curve. In general a rise (fall) in a competitor's price will shift the Bertrand demand curve to the right (left). If the slope is unchanged, profit will be increased by raising (lowering) price. If, however, the slope of the Bertrand demand curve is sufficiently reduced by the rise (fall) in the competitor's price, it may profit the price follower to change his price in the opposite direction. A market situation where this is likely is one in which small price differentials between the two producers will not materially affect the distribution of the market, while larger price differentials will send customers in large numbers to the producer charging the lower price. If we represent this situation by writing the slope of the Bertrand demand curve as a decreasing function of the absolute price difference between the two producers, we can show mathematically that a rise in the price leader's price will induce a fall in that of the price follower if they start from a certain likely range of prices, and vice-versa.

FRANCIS E. MCINTYRE, Cowles Commission for Research in Economics and Colorado College, *The Effect of the Undistributed Profits Tax upon the Distribution of Corporate Earnings—A Statistical Appraisal.* —Professor McIntyre pointed out that the undistributed-profits tax is not new in American legislative history, its essential features having appeared in income-tax legislation of the Civil War period. Its enactment in 1936 was apparently motivated by several not entirely consistent objectives: The prevention of personal-income-tax avoidance; the discouragement of extensive corporate saving and reinvestment; the securing of increased tax revenue from corporations, the greater equali-

zation of the tax burden on corporate and unincorporated enterprise, and the compulsion of larger dividends for small stockholders. The original Administration proposal would have abolished direct corporate-income taxation, imposing relatively high rates on undistributed earnings. After Senate modification the act retained the direct corporate-income levy, plus a surtax on undistributed earnings at lower rates than the Administration had proposed. It has generally been assumed by both advocates and opponents of this legislation that it resulted in a great increase in the proportion of earnings paid out by corporations in the form of dividends. A statistical analysis, based upon the Cowles Commission earnings and dividend indexes[1] for the period 1871–1937 reveals that, for all corporations included in these extensive indexes, the percentages of earnings distributed in 1936 and 1937 were near the average for the last 67 years. Twenty of the 67 years saw dividend-earnings ratios higher than 74 per cent—the 1936 figure. The 1937 percentage was 78. There were fourteen years, more prosperous than 1936, and without any undistributed-profits tax, in which the ratio of dividends to earnings exceeded that of 1936. In brief, the law did not cause a substantial change in corporate dividend policy. The most important single factor determining the ratio of dividends to earnings is clearly the general level of business prosperity. The less prosperous business is, the higher the ratio will be. The undistributed-profits tax could not, a priori, satisfy both the seekers of increased revenue from corporate taxation and the "over-saving" school which wanted to reduce corporate saving. Evidently the law accomplished very little toward increasing relative dividend disbursements. The recent American experience must be evaluated in terms of the revenue-producing efficiency of the enacted legislation.

On Thursday morning, December 29, Consumer Behavior was considered. Professor John H. Cover of the University of Chicago presided, and the following papers were presented:

H. G. Lewis and Paul H. Douglas, University of Chicago, *Some Problems in the Measurement of Income or Budgetary Elasticities.*—Mr. Lewis and Professor Douglas stated that, in the ideal situation (prices and tastes constant), the expenditures of consumer units (families, individuals, or institutions) on various groups of commodities and services may be considered as functions of income alone. These functional relations may be written in the form

$$x_i = f_i(Y)$$

where x_i is the total expenditure of the consumer unit on the com-

[1] *Common Stock Indexes, 1871–1937*, by Alfred Cowles 3rd and Associates, Principia Press, Bloomington, Indiana, 1938.

modity group i, and Y is the total income of the consumer unit. Certain derivative concepts and relations are basic to the theory and inductive analysis of these expenditure functions; they include the notions of *average expenditure*, x_i/Y, *marginal expenditure*, dx_i/dY, the *budgetary* or *income elasticity*, $\frac{d \log (x_i/Y)}{d \log Y}$, and the *elasticity of marginal expenditure*, $\frac{d \log (dx_i/dY)}{d \log Y}$. In general, the more urgent the expenditure on a commodity group, the larger $f_i(0)$, and the smaller the budgetary elasticity and the elasticities of average and marginal expenditure. In selecting the type of curve to fit to actual expenditure-income scatters, it is important to understand the implications of the types usually fitted. For example, the linear form, $x_i = a_i + b_i Y$, used by Allen and Bowley, and others, is subject to the severe restrictions: (1) It implies constant marginal expenditures, i.e., that the increments of expenditure on a commodity group associated with constant increments to income are the same irrespective of the income level. This means that redistributions of income would effect no change in the distribution of expenditure between the various commodities. (2) The budgetary elasticity must increase for "necessities"—those commodities with budgetary elasticities less than unity—and decrease for luxuries—those with budgetary elasticities greater than unity. The constant elasticity curve, $x_i = a_i Y^{b_i}$, is subject to fewer limitations; however, it is desirable to fit even less restricted types. The actual inductive analysis of the expenditure curves must also consider two other problems: (1) That consumer units observed vary more or less in homogeneity. (2) That the curves may be of varying degrees of short or long run.

ELIZABETH W. GILBOY, Harvard University, *The Measurement of Demand or Consumption.*—Dr. Gilboy suggested that the field of demand or consumption is one peculiarly suitable to the attack of mathematicians, statisticians, and plain (or nonmathematical) economists. General equations for price-quantity relations, or a part of equilibrium theory, have been developed for static and nonstatic conditions. Statisticians have attempted to examine these general equations empirically and to derive theoretical demand curves from price-quantity data. Plain economists have questioned and criticized these results with respect to their economic meaning. The result is a book such as that of the late Professor Schultz in which every mathematical and statistical step in the derivation of demand schedules is subjected to critical economic interpretation. It is now clear that approximate demand schedules in the theoretical sense can be derived statistically only from price-quantity data for agricultural commodities on the basis of a num-

ber of definite economic assumptions. The measurement of demand is thus definitely limited from this point of view. The possibilities of using income-expenditure data have just begun to be explored and may well offer econometricians a more promising opportunity for the measurement of demand. Allen and Bowley may be singled out as economists who have set up a simple statistical method for determining income-expenditure relations, income-expenditure elasticities, etc., and have related the process to equilibrium theory. Lough has attempted to examine income-expenditure relations and price-expenditure relations over time. These experiments indicate the type of data which should be investigated more thoroughly. In so doing one should not neglect the possibility of hypotheses other than those of equilibrium theory. Engel's laws and hypotheses like Keynes' propensity to consume should be examined empirically. It is essential that mathematicians, statisticians, and plain economists work together more closely. Assumptions implicit and explicit in mathematical and statistical processes must be examined continually as to their economic meaning. If this is done there seems good reason to believe that the field of consumption need no longer be left to psychologists and those outside the main development of economic thought.

ABRAHAM WALD, Cowles Commission for Research in Economics, *Criteria for a Constant Preference Scale Expressed in Terms of Engel Curves.*—Dr. Wald denoted the consumer's goods by $G_1, \cdots, G_n$. A set of goods q consisting of q_1 units of $G_1, \cdots, q_n$ units of G_n can be represented by a point q in the n-dimensional Cartesian space with the coordinates $q_1, \cdots, q_n$. The scale of preferences in this quantity space can be described by a real function $I(q)$ such that $I(q) = I(\bar{q})$ if q and $\bar{q}$ are two indifferent sets of goods, and $I(q) > I(\bar{q})$ if q is preferred to $\bar{q}$. Such a function is called an indicator. We restrict ourselves to the consideration of a region T of the quantity space in which the indicator can be approximated by a polynomial of the second degree. Then in each price situation the quantities purchased $q_1, \cdots, q_n$ are linear functions of the income E, that is to say

$$q_i = \alpha_i E + \beta_i \qquad (i = 1, 2, \cdots, n),$$

where α_i and β_i denote certain constants. Each of these equations can be represented by a plane curve, one axis representing the income and the other axis representing the quantity purchased. These curves are called also Engel curves. To each price situation corresponds a system of n Engel curves. The coefficients $\alpha_1, \cdots, \alpha_n$ are called the slopes, and the coefficients $\beta_1, \cdots, \beta_n$ the intercepts of the Engel curves. The solution of the following problem has been given: What are the necessary and sufficient conditions which must be satisfied by the Engel curves

observed in k different price situations in order that an indicator should exist, such that the theoreti Engel curves, derived from that indicator for the price situations considered, should coincide with the Engel curves observed? It has been shown that the necessary and sufficient conditions consist of a certain system of equalities and inequalities which must be satisfied by the slopes and intercepts of the Engel curves in the k different price situations considered. These conditions allow us to check to a certain extent by means of observable data whether we can assume that the scale of preferences remains unchanged during a longer period of time, or whether different groups of individuals have the same scale of preferences.

HORST MENDERSHAUSEN, Cowles Commission for Research in Economics, *The Relationship between Income and Savings of American Metropolitan Families.*—Dr. Mendershausen said that the Study of Consumer Purchases, undertaken by several agencies of the Federal Government in 1935–36, makes it possible to establish the relationship

TABLE 1

Data	Parameters			Income at break-even point in $1000[a]	Income elasticity of savings at	
	a	b	c		2% savings point[b]	an income of $10,000
New York	11.3	0.3	−27.2	2.29	6.33	1.47
Chicago	13.5	1.4	−25.5	1.62	9.28	1.66
Weighted average of metropolises[c]	13.6	0.5	−26.6	1.83	7.83	1.48
Weighted average of large cities[d]	10.6	2.0	−19.7	1.46	8.58	1.76
Weighted average of metropolises and large cities	12.8	1.2	−23.0	1.57	8.49	1.63
Brookings[e]	15.8	1.5	−25.0	1.40	10.22	1.62

(Certain errors in the mimeographed table distributed at the session have been corrected.)

[a] Where curve shows no change in assets and liabilities.
[b] Where curve shows 2% of income saved.
[c] Based on data for New York and Chicago.
[d] Based on data for Providence (R.I.), Columbus (Ohio), Omaha (Nebr.), Portland (Oreg.), Denver (Colo.), and Atlanta (Ga.). Weights: Number of native white families in city.
[e] Data for "nonfarm families," based on budget studies undertaken between 1918 and 1930. (*America's Capacity to Consume*, 1934, pp. 68, 72 and 79.)

between yearly income and savings—or deficits—of native white nonrelief families. In this analysis, use was made of the data for 8 cities with population over 150,000 in different regions of the United States. A skew hyperbola of the form

$$y = a + bX + c/X,$$

where X represents family income in thousands of dollars and y the amount of net change in assets and liabilities (Y) in per cent of family income, was found to be a very satisfactory simple mathematical expression for the sequences of scatter points between $500 and $20,000 of income. For higher income strata the curve type has to be modified. Table 1 contains some characteristics of several of the relationships obtained. The income elasticity of savings ($\eta = \partial Y/\partial X \cdot X/Y$) is larger than 1 everywhere, but decreases if we move from the break-even point to an income of $10,000. The rate of decrease itself decreases. There is an astonishingly high degree of similarity between the relationship found for metropolises and large cities in 1935–36 and that obtained from the Brookings data.

On Thursday afternoon the general subject was the Pure Theory of Industrial Production. Professor Joseph A. Schumpeter of Harvard University was chairman and the papers were as follows:

IRVING FISHER, Yale University, *A Three-Dimensional Representation of the Factors of Production.*—Professor Fisher exhibited a cardboard model of the factors of production. This representation has a double usefulness. It helps the classroom student to see, literally to see with his eyes, the truth of the following four propositions, which are not easy for him to see in any other way. (1) A law of diminishing returns may exist for each separate factor of production while at the same time a law of proportionality may hold true for all the factors taken together in a composite. (2) If this is the case, the remuneration of each factor according to its marginal contribution to the product will leave no residual unaccounted for, but if this is not the case there must be a residual. (3) Given a law of proportionality for all factors combined and a law of diminishing returns for one factor or group of factors alone, it follows that the remaining factors grouped in a composite must be subject to a law of diminishing returns; moreover that law must be of a certain specified type as determined by the hypothesis just stated. (4) Under the same conditions either of the two laws of diminishing return applies without change, no matter what the assumed magnitude of the other co-ordinate factor or group may be. The other usefulness of this three-dimensional construction is in helping clarify the thoughts of economic thinkers themselves; for correct visual pictures usually yield the clearest concepts.

JAKOB MARSCHAK, University of Oxford, *Measuring the Mobility of the Factors of Production.*[2]—Dr. Marschak suggested the following definition of mobility (λ_{ab}) of a factor of production:

[2] See *Studies in Mobility of Labour*, by H. Makower, J. Marschak, and H. W. Robinson, Oxford Economic Papers No. 1, Oxford University Press.

$$\lambda_{ab} = \frac{M_{ab}}{ab} : T',$$

where M_{ab} = annual net migration of the factor from region, or industry, A to B,
a and b = amounts of the factor already available in A and B,
T' = discrepancy in reward rates defined as

$$(p_a - p_b), \text{ or as } \frac{(p_a - p_b)}{p_a}, \text{ or as } \frac{(p_a - p_b)}{(p_a + p_b)},$$

where p_a and p_b are the reward rates in the two regions. Further parameters are introduced, to study the influence of distance; the time lag between reward discrepancy and migration; and the relative weights of "push" and "pull." Thus, a fuller definition of mobility (μ_{ab}) would be

$$\mu_{ab}{}^{(t)} = \frac{M_{ab}{}^{(t)}}{ab} : T^{(t-\theta)},$$

where the indices in parentheses indicate the time points to which the variables refer, and T is defined as, say,

$$(p_a - kp_b)/p_a D^{\nu};$$

D being the distance, k the "weight of the pull," and ν the "elasticity of migration with respect to distance." For studying interregional labor mobility in Great Britain 1927–1936 we put $k = 1$ in the first approximation and obtained, for 40 counties and Oxford, $\theta = \frac{1}{2}$ to 1 year, $\nu =$ 1.5 to 2.0 (with correlation coefficients about 0.85), when p_a and p_b were represented, not by expected earnings (including or excluding the dole) but by employment rates. This use of employment rates (suggesting that the relevant psychic reward is the chance of getting a job rather than of increasing the wage rate) was justified by the actual correlation, for selected areas, of net migration figures on the expected earnings and on the employment rates: the rank correlations, for seven investigated large areas, between net migration and (1) employment rate, (2) expected earnings, (3) expected earnings and dole, were respectively 0.97, 0.75, 0.11. To evaluate these, interregional wage differences within an industry had, however, to be neglected. The material used is of two kinds: (α) one-time counts of migrants, (β) annual net migration gains or losses. Material (α) gives—after correcting for deaths, etc.—migration by pairs of regions, thus permitting distance analysis; but, being cumulative, this material forbids time studies. The reverse is true of material (β). There is also material (γ) giving annual interregional movements, but it is unfortunately less reliable. The value of μ_{ab} seems to fall after the slump. The comparison of a sample of

Welshmen in Oxford with the unemployment structure in Wales suggests, as a cause, the prevalence of long-period unemployed in the slump; these unemployed are less able to migrate. The study of the variations of μ_{ab} as between various pairs of regions leads to the analysis of their industrial structure and must be checked by direct analysis of inter-industrial mobilities. Materials analogous to (β) and (γ) are available in Britain for this purpose. Similarly, data on profits and net investments by industries may be used—in analogy to (β)—for a study of capital mobility.

PAUL A. SAMUELSON, Harvard University, *A Restatement of the Theory of Cost and Production with Reference to its Operational Aspects.*—Mr. Samuelson remarked that in standard economic works there is often no clear and correct statement of the relationship between the production function and the cost curve; even in the few places where a reasonably correct account of the conditions of equilibrium are to be found, there is no attempt to derive from these conditions theorems of *operational* significance. The conditions of equilibrium imposed from within the firm by the attempt to maximize profits require (1) that any output which is to be produced must be produced at the lowest possible total expenditure; (2) that out of all possible outputs, that output will be selected which maximizes profits. Under proper assumptions of continuity, the first condition requires that all factor prices, taken as given by the firm, be equal to marginal cost times marginal physical productivities. The second condition requires that marginal cost be equal to marginal revenue. In the general case, including continuity as a special instance, the conditions of equilibrium can be simply stated in terms of finite inequalities. These are easily shown to yield unambiguous, meaningful restrictions upon price-quantity behavior. The condition that a firm earn no profits cannot be deduced from any internal condition of equilibrium, but is realized, if at all, only through the operation of free entry. The relation of this to homogeneity of the production function, and to Walras' marginal productivity theorems, is discussed.

Later in the afternoon of Thursday a special session was held in memory of Professor Henry Schultz of the University of Chicago, who, with his family, was killed in an automobile accident near San Diego, California, on November 26, 1938. Professor W. L. Crum of Harvard University presided and the following addresses were made:

HAROLD HOTELLING, Columbia University, *The Work of Henry Schultz.*

PAUL H. DOUGLAS, University of Chicago, *Henry Schultz as Colleague.*

These are printed in full in this issue of ECONOMETRICA.

A session on Mathematical Statistics was held Thursday evening. The chairman was Professor Irving Fisher of Yale University, and the following papers were given:

J. NEYMAN, University of California, *On the Hypothesis of Linearity of Regression in Economic Research.*—Professor Neyman discussed the results obtained by Miss H. V. Allen.[3] Her study was suggested by papers at the Oxford meeting of the Society in 1936, in which the authors considered systems of random variables $x_1, x_2, \cdots$ each having more or less the same structure, like the following:

$$x_i = a_i\xi + \alpha_i, \tag{1}$$

where a_i denotes a constant coefficient, and ξ and the α's random variables supposed to be mutually independent. In these papers the regression of any x on any other x was assumed to be linear. This was later questioned by Professor Frisch, and Miss Allen was advised to find the conditions to be satisfied by the variables ξ and α_i, to assure such linearity of regression, whatever the values of the coefficients a_1. She found, if ξ and α_i are known to possess all the moments, then a necessary and sufficient condition for linearity of regression is that ξ and the α_i's should be normally distributed. Under such conditions the x_i's would follow a normal law. Since normal distributions are seldom met with in economic and social studies, mathematical models invented to represent phenomena involving variables like the x_i's must renounce either the hypothesis of independence of ξ and the α's or that of linearity of regression. In some cases deviation from linearity will be negligible, but, in development of theory which depends on the shape of the regression line, Miss Allen's results must be taken into account. Moreover, apart from being a warning to authors engaged in mathematical treatment of economic problems, her theorem describes another outstanding property of the normal law, similar to the one recently obtained by P. Lévy and H. Cramér:[4] if x_1 and x_2 are independent, then for their sum x_1+x_2 to be normally distributed, it is necessary and sufficient that both x_1 and x_2 should follow a normal law. The speaker's colleague, Mr. H. Wilenski, has since proved Miss Allen's results on somewhat less restrictive conditions about the moments, and it is hoped that his results will soon be published. Also Dr. S. Kolodziejczyk gave an example in which the regression of x_2 on x_1 is linear without α_1 and ξ being normally distributed, but being assumed to follow the Cauchy law of frequency, all moments of which are divergent. It

[3] "A Theorem Concerning the Linearity of Regression," *Statistical Research Memoirs*, Vol. 2, December, 1938, pp. 60–68.

[4] H. Cramér, *Random Variables and Probability Distributions*, Cambridge University Press, 1937.

follows therefore that, for the validity of the statement, some restriction concerning the moments is necessary. As all the variables considered in economic studies are of necessity bounded, and, therefore, have finite moments, Kolodziejczyk's example, although of considerable theoretical interest, does not diminish the relevance of Miss Allen's results from the point of view of the work in economics.

HAROLD HOTELLING, Columbia University, *The Selection of Variates for Use in Prediction.*—Professor Hotelling pointed out that when a variate y is to be predicted by a regression equation, there are frequently more independent variates available for use in this equation than can expediently be used. One criterion of choice among the available variates is to select the set having the greatest possible multiple correlation with y. This, however, raises the question of the significance of the excess of this maximum multiple correlation over the other possible multiple correlations. No satisfactory solution of this problem appears possible if the usual assumption is made that all the variates have jointly the multivariate normal distribution, since the standard errors of the differences of the multiple correlations in the same sample involve unknown parameters, and since the distributions are not exactly known. But upon assuming the values of the available independent variates $x_1, \cdots, x_p$ the same for all the samples with which the known one is to be compared, while only y varies from sample to sample, these difficulties are to a considerable extent removed, and exact solutions for the leading problems at least become available. If only one of these p variates is to be used, the exact test reduces to an application of the variance ratio distribution. If it has been decided to use certain variates and the choice of one more among several is to be made, the same test may be applied merely by replacing the correlations by partial correlations and reducing the number of degrees of freedom by the number of variates already chosen. But there are further mathematical problems in this field which have not yet been solved.

WASSILY LEONTIEF, Harvard University, *Decomposition of Time Series on the Basis of Non-Correlation Principle.*—Professor Leontief showed that any dependence between the separate components of an empirical time series can be "artificially" produced through appropriate choice of the method of "decomposition," i.e., in any particular instance such correlation must be considered to be the result of the particular choice of statistical procedure. It follows that the very existence of *significant independence* between theoretical components of an empirical time series implies a possibility and *necessity to eliminate it so as to reduce to a minimum the incidence of operational conventions.* As the two steps—(a) the preliminary decomposition of the given

series and the subsequent (b) computation and elimination of significant interdependence—are *strictly complementary*, a statistical procedure which satisfies all independence requirements *uno actu* appears to be preferable. So long as economic theory treats the concepts of the trend, business cycle, etc., as index functions, i.e., allows any kind of simple (monotone) transformation of their numerical measures, any attempt to *demonstrate* statistically the existence of some kind of quantitative *interdependence* between the separate components of a time series is *logically impossible*. In the practical application of the independence principle the least-square method as used in trend fitting can be considered to be a special case of such application: Using the Pearsonian coefficient of correlation as a measure of "dependence" and postulating that the deviations from the "trend" have to be noncorrelated with the trend itself or any of its derivatives, it is possible to determine the coefficients of the trend polynomial; they turn out to to be identical with those derived by the least-square method.

On Friday morning, December 30, Economic Problems were discussed, with Professor Joseph A. Schumpeter of Harvard University in the chair. The papers were as follows:

ABBA P. LERNER, London School of Economics, *Equilibrium and Dynamic Concepts in the Theory of Employment.*—Mr. Lerner said that the recently developed theory of employment associated with Mr. Keynes can be usefully described as more *dynamic* than the "classical" theories of equilibrium that left no place for unemployment. It is, however, incorrect to suppose that this is because it brings in *expectations* in using concepts like the marginal *efficiency* of capital rather than the marginal *productivity* of capital. Expectations were always fundamental in "classical" economic theory and what distinguishes Mr. Keynes' marginal efficiency of capital is rather the stress on the *rate of investment* instead of on the *quantity of capital* in clarifying the *dimensions* involved, so that it should really have been called the marginal efficiency of *investment*.[5] The theory is more dynamic in that it develops a *shorter-period* analysis and so enables a *path* to be ininvestigated that may or may not lead toward the long-period equilibrium position. The path consists of positions of short-period equilibrium, the determinants of which change in the longer period. The theory is not as dynamic as some would wish because it *assumes* the short-period equilibrium to be reached, and a failure to recognise the (short-period) equilibrium nature of the analysis has been responsible for unhelpful criticisms of concepts like the multiplier or the liquidity-preference schedule. It would indeed be better to have a *more* dynamic

[5] See A. P. Lerner, "Capital, Investment and Interest," Manchester Statistical Society, Group Meetings, 1936–37.

analysis (which would have to run in terms of a still-shorter-period equilibrium) but such analysis has not yet been developed in usable form. *Dynamic* paths are inevitably discontinuous, consisting of points where some shorter-period equilibrium or adjustment is reached. They should not be confused with *planned* paths which can be conceived of as strictly continuous because the discovery and correction of error does not enter. However, Harrod and Kalecki have been able to develop other dynamic aspects of the theory of employment.

ROLLIN F. BENNETT, Columbia University, *On the Use of Mixed Difference-Differential Equations.*—Mr. Bennett remarked that the form in which the solution of a system of mixed difference and differential equations is ordinarily given—an infinite series of cyclical components with coefficients dependent on boundary conditions—leaves much to be desired. A more useful form of solution may be found by formulating the system explicitly, instead of implicitly, as a problem in the propagation of impulses. By introducing an impulse sequence explicitly as an independent function of time into the system of equations, and then solving for each dependent variable explicitly in terms of the independent variable, information about the properties of the system becomes more readily obtainable. Moreover, if the impulse variable can be interpreted as an observable datum, then the solution can be used to answer very practical questions—such as how a given sequence of public-works expenditures will modify the time shape of national income. If, however, the impulse is interpreted as a random variate, then the solution, with slight modification, can be used directly to predict the future of a variable in terms of its own past. The form of solution is in every case a linear functional, with the kernel of the integral summarizing the most important intrinsic properties of the system. Symbolically, the solution may be represented as a ratio of two determinants operating on the independent variable; and the method of calculating the inverse of the operation represented by the determinant in the denominator presents the problem of chief interest.

LOWELL J. CHAWNER, United States Department of Commerce, *The Residential Building Process: An Analysis in Terms of Related Influences.*—Mr. Chawner reported briefly upon an analysis of the behavior of residential rents and residential building activity in terms of related economic and other social influences over the period from 1914 through 1937 in nonfarm areas in the United States. The results and some of the technical details of this analysis will be available in a forthcoming study which the speaker has prepared for the National Resources Committee, and which will be published by that agency. An a priori analysis indicated that the trends in rents are logically a function of changes in the total number of families to be housed, in the

number of available family units, in the trends in family income, and in the competing claims upon family income of other items of consumption, particularly those of relatively inelastic demand. Using measures of these various items, a linear equation in 3 independent variables, fitted by algebraic methods, was derived which accounts for 98 per cent of the variation in rents over this period (1914–1937). The independent variables used were: an index of income per nonfarm family, a ratio of the number of families to the number of available units, and a ratio of the price of food to the price of other items in cost of living other than food and shelter. The latter is the only competing item of expenditure for which suitable measures are available which appears to be significantly related to rent changes. The number of new residential units started in a given year appears to be influenced principally by annual increments in the number of families in nonfarm areas, the average income of nonfarm families, the cost of ownership of houses (particularly building costs, interest rates, and taxes), and by the condition of the market as measured by rents and the percentage of occupancy. These influences were expressed in terms of 3 independent variables. The first is an index of the annual increments in the number of families in nonfarm areas in the United States, the second a ratio of family income to the annual cost of ownership, the third the product of an index of rents and the percentage occupancy each year. A measure of the cost of ownership was calculated by multiplying the index of building cost by a weighted sum of interest plus taxes. Using these 3 independent variables an equation of semilogarithmic type was fitted to the number of residential units upon which construction was started in the United States, annually, over the period from 1914 through 1937. This equation, derived by algebraic methods, accounts for 93 per cent of the variation in the number of units built annually during these years, a period of unusually wide fluctuations in activity. Particular attention should be called to the need for more representative statistics of vacancies and improved demographic statistics (especially marriages, divorces, and deaths) as the basis for the development of studies of behavior of residential building in connection with practical business problems and in the formulation of public policy relating to this large and important branch of economic activity.

The final session was held on Friday afternoon. The subject was Monetary Processes, and Professor Francis E. McIntyre of the Cowles Commission for Research in Economics and Colorado College presided. The following papers were presented:

JAMES HARVEY ROGERS, Yale University, *Absorption of Credit vs. Absorption of Savings.*—Professor Rogers' paper was a continuation of several earlier ones, the first of which was published in Volume 1, Num-

ber 1 of ECONOMETRICA. He proposed to define and discuss: (1) the absorption of credit, (2) the absorption of savings, (3) certain interconnections between the absorption of credit and the absorption of savings, and (4) several significant examples of these interconnections. The absorption of credit must be traced on two levels: (a) on the level of member-bank reserves, (b) on the level of central-bank reserves. Once balance sheets of our money system are set up on both these levels, the total expansion possibilities of our banking system under various hypotheses can be estimated. A table of such computations, which have been recently made, was presented. In general, the absorption of credit comes through the expansion of loans and investments of the commercial banks. Specifically, it comes about through the expansion of deposits, through the flow of cash to circulation, and through adverse changes in the monetary base, of course. Besides the absorption of credit, there is, of course, also the *dissipation* of credit. This occurs primarily through bank failures and the consequent scaling down or obliteration of bank deposits. Savings are defined as that part of income withheld from spending. In the investment and margin-loan policies of banks, there is a substitution of credit absorption for savings absorption. In the relending of idle deposits by individuals and companies, there is an effective increase in savings in so far as security purchases are concerned without even any further absorption of credit. In conclusion, Professor Rogers pointed out the great substitution of credit absorption for savings absorption in the late twenties and commented upon the possibility of similar substitution in the near future should other conditions become favorable.

HAROLD T. DAVIS, Northwestern University and Cowles Commission for Research in Economics, *Some Lessons from the Equation of Exchange.*—Professor Davis presented a systematic statistical study of the meaning of the equation of exchange, $MV+M'V'=PT$, where M is circulating money, M' circulating credit, V and V' their respective velocities, P price, and T trade. The equation was assumed to bear a relationship to economic theory similar to that borne to the theory of electrical conduction by the equation $E=IR$, where E is the electromotive force, I the current, and R the circuit resistance. The tautological character of the equation was shown to disappear when independent measures are set up for the variables. The theorem of Carl Snyder as formulated by E. V. Huntington[6] was submitted to a statistical test and some anomalies pointed out and explained. It was discovered that MV is approximately equal to 12.5 times national income and this relationship was used to compute the velocity of credit

[6] ECONOMETRICA, Vol. 6, 1938, pp. 177–179.

over the period 1912–1919 for which no estimates appeared to be available. The paper concluded with a study of the Spanish economy between 1500 and 1650 by means of data recently published by E. J. Hamilton. Estimates were made of the amount of treasure necessary to support the price structure of the Middle Ages and these estimates were found to agree with those of W. Jacob made quite differently many years ago. Spain was shown to have reached the top of its long trade advance by 1620 and a similarity was indicated between the Spanish economy of the subsequent period and that of present-day United States.

DICKSON H. LEAVENS

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ANNOUNCEMENT OF THE ELSINORE MEETING, AUGUST 26–28, 1939

THE NINTH European meeting of the Econometric Society will be held at Marienlyst, Elsinore, Denmark, from August 26 to 28, 1939.

Accommodations with full board will be available at a charge of 14, 12, and, for a limited number of members, 10 or 11 Danish Kroner per day (exclusive of 15 per cent for service). Members desiring to attend the meeting are requested to write before July 15 to Professor F. Zeuthen, Hostrups Have 1, Copenhagen, Denmark. They will receive in reply further details regarding programme, accommodations, trains, etc. The early date of the meeting this year has been chosen in order to enable the Scandinavian economists, whose university year begins September 1, to be present.

The first day of the meeting is intended to coincide with the Fourth Nordic Meeting of Economic Research, which will be held in the same place from August 24 to 26. The programme for the 26th is going to deal with subjects of common interest to both meetings such as the "Stockholm School" and The Theory of Production, Cost and Price Policy. The econometric programme for the next days will depend on proposals received from members.

Drafts or summaries of papers should be sent before July 1st to Professor Ragnar Frisch, Vinderen, Oslo, Norway.

INSTRUCTIONS TO CONTRIBUTORS REGARDING PAPERS FOR ECONOMETRICA

1. An author, in presenting a manuscript to ECONOMETRICA, guarantees that this paper will not be published elsewhere before its publication in ECONOMETRICA.

2. Manuscripts should be in English or French, preferably English. Manuscripts from authors in the United States and Canada should be sent to the managing editor of ECONOMETRICA, 301 Mining Exchange Building, Colorado Springs, Colorado, U. S. A. Manuscripts from authors in other countries should be sent direct to the editor, Professor Ragnar Frisch, University of Norway, Oslo, Norway. Manuscripts should be typewritten, double-spaced; complicated formulae, however, being filled in by hand.

3. In the case of manuscripts which the editor finds too long (for example, because of the inclusion of extensive tables of data or details of computation of interest only to a limited number of specialists) but otherwise acceptable for publication, arrangements may be made by which the article may be published in an abridged form or without the supporting tables; the complete manuscript or the tables will be deposited with the American Documentation Institute, Washington, D. C., and a note published with the article that any interested may obtain from the Institute a complete photographic reproduction of the material, in the form of either 35-mm. microfilm at 1 cent per page or full-size photoprints at 10 cents per page, plus a service charge of 20 cents.

4. If a paper is to be illustrated by charts, the author should submit either (a) drawings complete with lettering suitable for direct reproduction; (b) drawings with legends and scales lettered lightly in blue pencil, so that printed letters and figures can be pasted on; or (c) rough sketches with tables of data from which final charts can be prepared. In the case of (b) or (c) the cost of preparing the drawings in final form will be charged to the author. Drawings should be made at least twice the size in which they are to be reproduced, and should be planned for reduction to a final size which will fit the proportions of the page and take up no more room than is needed. Lines and lettering to be reproduced should be in black drawing ink on white ground, or, if more convenient, on *blue-ruled* co-ordinate paper, the lines of which will not be reproduced in the photo-engraving.

5. Two galley proofs together with original manuscript, will be mailed by the printer direct to the author. Within ten days the author should forward one corrected galley proof *and the manuscript* to the managing editor, Dickson H. Leavens, 301 Mining Exchange Build-

ing, Colorado Springs, Colorado, U. S. A. This applies to all galley proofs, whether from authors in the United States, or Canada, or other countries. Revised galley proofs will be sent to the author only upon request, and may delay publication.

6. If the author, after receiving the galley proof, makes alterations in his original manuscript, the cost of incorporating such alterations, if it exceeds $2.00, will be charged to the author. In this connection it should be noted that often the insertion or deletion of even one word will make necessary the resetting of several lines. Expense can be saved by taking care that inserted words occupy the same amount of space as those deleted.

7. Each author will be furnished, free of charge, with fifty reprints of his paper, without covers. The printer, when mailing the galley proof to the author, will inclose an order blank in case the author wishes to order additional reprints. The schedule of charges for reprints is as follows:

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